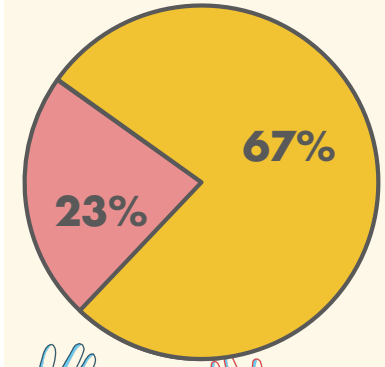
1. Majority Rules

Big Q: In an election with two candidates, how do we fairly decide a winner?

Example

Suppose Alice and Bob are running for mayor of a town with 100 residents.

The votes come in and they look like this:



	Alice	Bob
Number of votes	67	23

Q: Who should win, and why?

A: Alice should win because she got more votes.

However there are other ways to decide a winner!

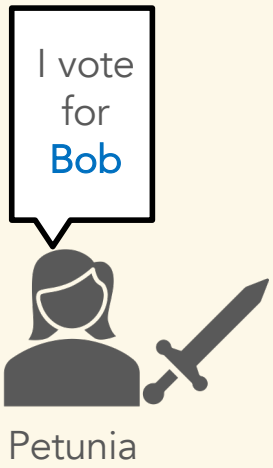
Petunia is dictator

One of the residents in town is named Petunia, and let's say she voted for Bob.

I claim that the way we're going to decide the election is that *Bob will win because that's who Petunia voted for.*

This is an example of a voting system! It's a way to decide the outcome of an election from the ballots of voters.

Let's call this the "**Petunia is dictator**" voting system. Here's how it works:



Petunia is dictator

1. Everyone submits their *ballots* indicating who they prefer for the election
2. We ask Petunia who she voted for
3. Whoever Petunia voted for wins

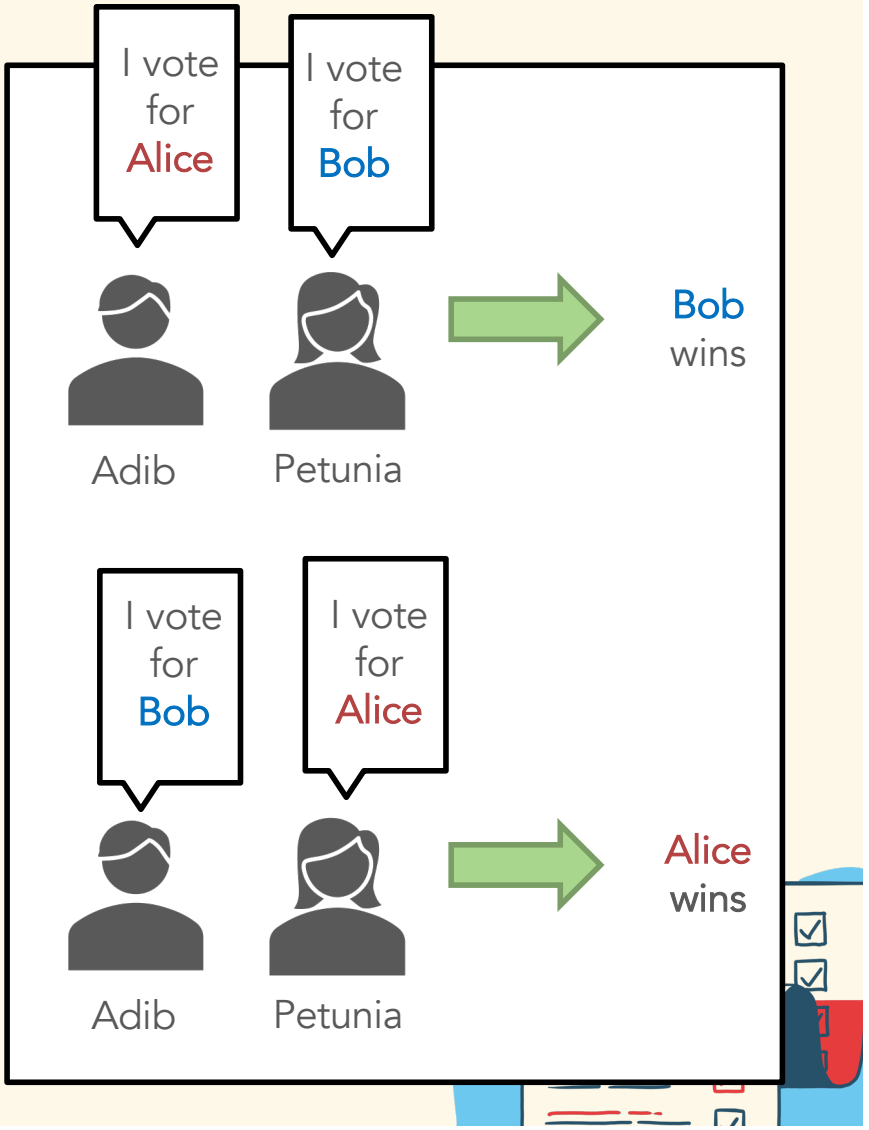
Q: Is this voting system *fair*? Why or why not?

Anonymity

Definition

A voting system for an election with 2 candidates is ***anonymous*** if it treats all the voters equally. That is, if any two voters traded votes, the outcome of the election would stay the same.

A: The **Petunia is dictator** voting system is not anonymous --- because if Petunia switched her vote with another resident, the outcome of the election would change.



Bob is king

Let's suppose that Bob has a crown on his head, and that he is king of the town.

Let's define a new voting system, called "**Bob is king**", where Bob wins no matter what.

Bob

Bob is king

1. Everyone submits their *ballots* indicating who they prefer for the election
2. Bob wins

Q: Is the **Bob is king** voting system *anonymous?*

A: Yes! If any two voters swap their votes, the outcome of the election doesn't change – Bob will still win.

Bob is king

Suppose *everyone* in town votes for Alice. Under the **Bob is king** voting system, Bob still wins.

	Alice	Bob
Number of votes	100	0

Bob is king → Bob wins

100 %

Definition

A voting system for an election with 2 candidates is ***unanimous*** if, when every voter votes for one candidate, then that candidate wins.

Q: Is this a fair voting system?

So the **Bob is king** voting system is not *unanimous*.

Unanimity

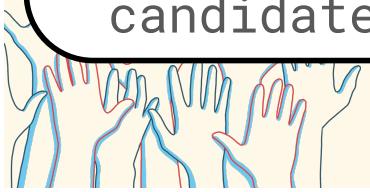
Definition

A voting system for an election with 2 candidates is ***unanimous*** if, when every voter votes for one candidate, then that candidate wins.

Q: Is the **Petunia is dictator** voting system *unanimous*?

A: Yes! If everyone votes for one candidate, then Petunia also voted for that candidate, so that candidate wins.

	Petunia is dictator	Bob is king
Anonymous	⊘	✓
Unanimous	✓	⊘



Comparing voting systems

This is how we are going to mathematically approach voting systems.

We will come up with ***voting criteria*** like "unanimity" or "anonymity" and then ask questions like:

- is this a reasonable property for voting systems to have?
- does the voting system have this property?
- etc.

	Petunia is dictator	Bob is king
Anonymous	⊘	✓
Unanimous	✓	⊘

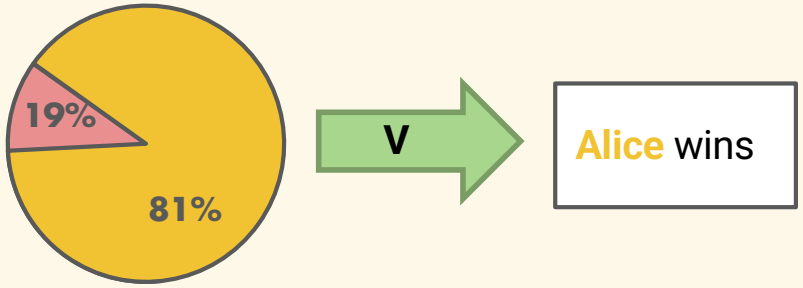
Once have a collection of nice properties, we can ask harder questions like:

- What can we say about voting systems that satisfy these criteria?

Neutrality

Let's suppose we have a voting system **V**, and I haven't told you how it works.

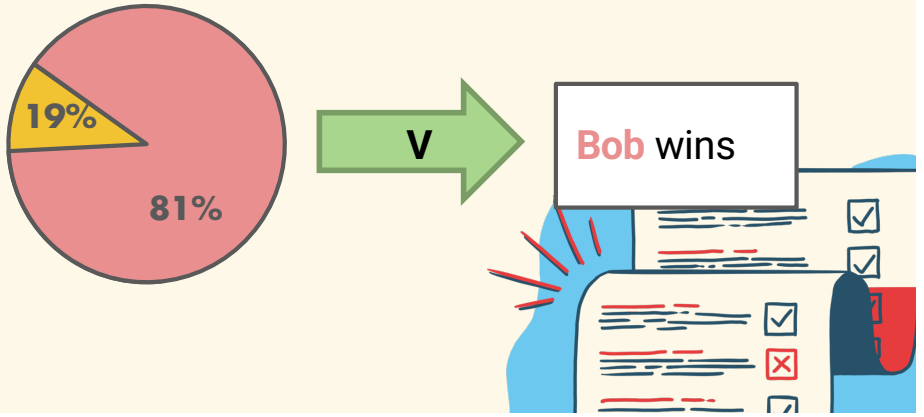
But in the following election, under voting system **V**, Alice wins:



Q: If we swap the colors red and yellow, what should happen?

A: A fair answer would be that now Bob should win.

Q: If every Alice voter switched their vote to Bob, and every Bob voter switched their vote to Alice, what should happen?

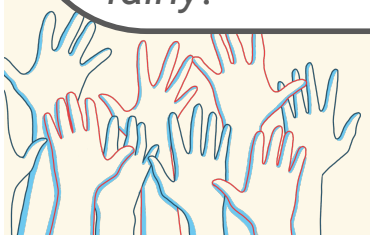


Neutrality

The types of voting systems that exhibit this behavior are called *neutral* voting systems.

The idea behind neutrality is that it shouldn't depend on *who* got which votes. That is if Alice wins with 81 votes, then Bob should win with 81 votes.

That is, neutral voting systems are ones which *treat candidates fairly*.



Definition

A voting system for an election with 2 candidates is ***neutral*** if, for any election, if candidate A wins and then everyone switches their vote, then candidate B wins.

Example: A voting system where "the incumbent wins ties" is not neutral.

Neutrality

Definition

A voting system for an election with 2 candidates is ***neutral*** if, for any election, if candidate A wins and then everyone switches their vote, then candidate B wins.

Q: Is the **Petunia is dictator** voting system *neutral?*

A: **Yes!** Suppose everyone casts their ballots and Alice wins (meaning Petunia voted for Alice). If everyone switches their ballot, then in particular Petunia switches her ballot to Bob, and now Bob wins.

Q: Is the **Bob is king** voting system *neutral?*

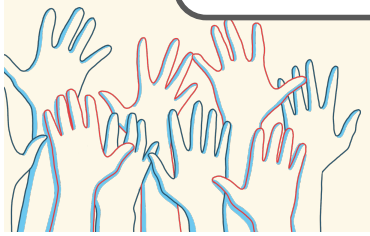
A: **No!** If everyone casts their ballots, then Bob wins. If everyone switches their ballot, then Bob still wins and the outcome of the election is unchanged.

Voting criteria

Let's update our chart with this new info:

	Petunia is dictator	Bob is king
Anonymous	⊘	✓
Unanimous	✓	⊘
Neutral	✓	⊘

We're going to talk about one more property of voting systems before returning to "majority rules"



Monotonicity

Again let's suppose we have an abstract voting system **V**, and I haven't told you how it works.

I do tell you that in the following election, Alice wins:

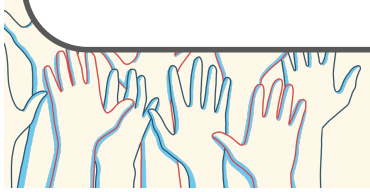
	Alice	Bob
Number of votes	67	23

V → Alice wins

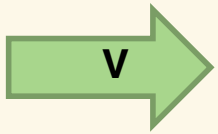
Now suppose we run the same election again, and all the Alice voters still vote for Alice. However, now *7 of the Bob voters switch their vote for Alice:*

	Alice	Bob
Number of votes	↑ 74	↓ 16

Q: Who should win in this new election?

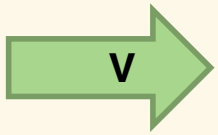


	Alice	Bob
Number of votes	67	23



Alice wins

	Alice	Bob
Number of votes	↑74	↓16

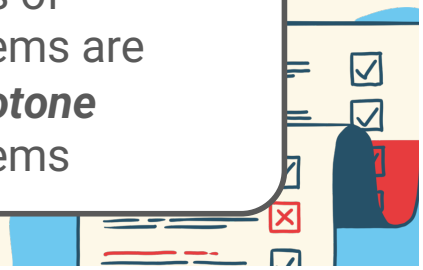


???

A: It makes sense that Alice should still win…right?

If Alice won with 67 votes, gaining any new votes shouldn't cause her to lose.

These types of voting systems are called ***monotone*** voting systems



Monotonicity

Definition

A voting system for an election with 2 candidates is ***monotone*** if it is impossible for a winning candidate to become a losing candidate by gaining a new block of votes.

As an exercise later, we should think about why **Petunia is dictator** and **Bob is king** are both monotone voting systems

For now, we're going to talk about a voting system that fails to be monotone.

Minority rules

We're going to define a new voting system called "**minority rules**." Basically, whoever gets the fewest votes wins.

	Alice	Bob
Number of votes	22	78

Minority rules → Alice wins

Minority rules

1. Everyone submits their *ballots* indicating who they prefer for the election
2. We count up all the ballots, and whoever got the *fewest votes* is the winner.

Suppose all the Alice voters stick with Alice, but now 40 Bob voters switch over to Alice. If **minority rules** were *monotone*, this shouldn't affect anything.

	Alice	Bob
Number of votes	↑ 62	↓ 38

Minority rules → Bob wins

Therefore **minority rules is *not monotone*.**

Majority rules

Finally, we get to the voting system we know and love: **majority rules**.

Majority rules

1. Everyone submits their *ballots* indicating who they prefer for the election
2. We count up all the ballots, and whoever got the ***most*** *votes* is the winner.

Q: Is **majority rules** *unanimous?*

A: **Yes!** If a candidate got *all of the votes*, then they definitely have the most votes, so they win.

Q: Is **majority rules** *monotone?*

A: **Yes!** If Alice got a majority of votes, then if she gains any more votes she will still have a majority!

Majority rules

Definition

A voting system for an election with 2 candidates is ***neutral*** if, for any election, if candidate A wins and then everyone switches their vote, then candidate B wins.

Q: Is **majority rules** *neutral*?

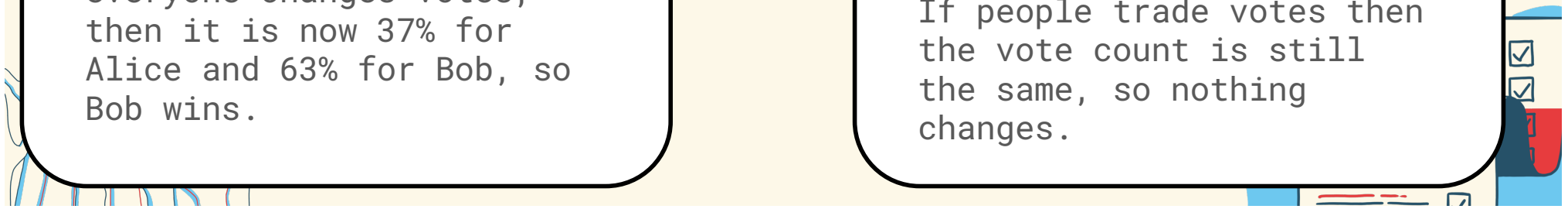
A: **Yes!** Suppose for example that the vote is 63% for Alice, and 37% for Bob. If everyone changes votes, then it is now 37% for Alice and 63% for Bob, so Bob wins.

Definition

A voting system for an election with 2 candidates is ***anonymous*** if it treats all the voters equally. That is, if any two voters traded votes, the outcome of the election would stay the same.

Q: Is **majority rules** anonymous?

A: **Yes!** Majority rules only cares about *how many* votes each candidate got, not about who cast the votes. If people trade votes then the vote count is still the same, so nothing changes.



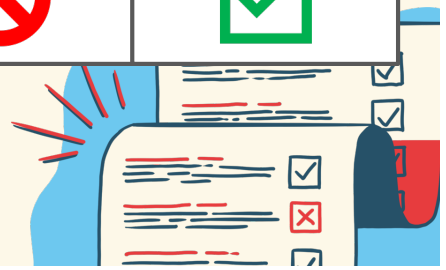
We came up with four pretty reasonable criteria for elections with 2 candidates:

- **anonymity**
- **unanimity**
- **neutrality**
- **monotonicity**

We see that **majority rules** satisfies all four!

Our criteria

	Petunia is dictator	Bob is king	Minority rules	Majority rules
Anonymous	⊘	✓		✓
Unanimous	✓	⊘		✓
Neutral	✓	⊘		✓
Monotonic			⊘	✓



May's Theorem

Q: Are there any other voting systems (besides majority rules) that satisfy:

- anonymity
- unanimity
- neutrality
- monotonicity?

In 1952, mathematician Kenneth May proved that the answer is no!

May's Theorem

Majority rules is *the only voting system* in an election between two candidates which is anonymous, neutral, monotone (and unanimous).

Kenneth May, 1952

Big idea: There is only one reasonable way to vote in an election with two candidates: majority rules.

Next time...

Big idea: There is only one reasonable way to vote in an election with two candidates: majority rules.

This is **not true** when there are 3+ candidates. Things get a lot more complicated and a lot more interesting.

Key Vocab:

- Voting system
- Voting criteria
- Majority rules
- May's theorem
- Anonymity
- Unanimity
- Neutrality
- Monotonicity

Exercises

Exercise 1: Is *minority rules* an anonymous voting system? Is it unanimous? Neutral? Argue why or why not.

Exercise 2: Come up with a voting system for elections with two candidates which is different than any of the ones we've discussed so far. Is it anonymous? Unanimous? Neutral? Monotone?

Exercise 3: Argue why Petunia is dictator and Bob is king are both *monotone* voting systems.