IWOAT SUMMER SCHOOL ON STABLE MOTIVIC HOMOTOPY THEORY, SPRING 2024

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1. Talk 2: Motivic spaces

Abstract. The construction of the unstable $\mathbb{A}^1$-homotopy category over a base. Functorialities, f_*, f^*. Thom spaces. Homotopy purity

1.1. Nisnevich sheaves.

Definition 1.1. We call a square

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xhookrightarrow[i]{} & X \end{array}$$

a *distinguished Nisnevich square* if p is étale, i is an open immersion, and p is an isomorphism over $X - U$.

Example 1.2. Let $\phi\colon R \to S$ be an étale ring map, and let $f \in R$ so that $R/f \xrightarrow{\sim} S/\phi(f)$ is an isomorphism of rings. This gives a distinguished square

$$\begin{array}{ccc} \operatorname{Spec}(S_{\phi(f)}) & \longrightarrow & \operatorname{Spec}(S) \\ \downarrow & \lrcorner & \downarrow \\ \operatorname{Spec}(R_f) & \longrightarrow & \operatorname{Spec}(R). \end{array}$$

Definition 1.3. We say that $F \in \mathrm{P}(\mathrm{Sm}_S)$ is *Nisnevich excisive* if it sends distinguished Nisnevich squares to homotopy pushout squares of spaces.

Theorem 1.4. (Voevodsky) We have that $F \in \mathrm{P}(\mathrm{Sm}_S)$ is a Nisnevich sheaf if and only it is Nisnevich excisive (see [AHW17, p. 3.2.5] to read about this).

This reduces checking descent on squares, of which there are a small number, hence we have that $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S) \subseteq \mathrm{P}(\mathrm{Sm}_S)$ is a reflective subcategory, admitting a left adjoint which we call L_{Nis}.

Example 1.5. Algebraic K-theory is a Nisnevich sheaf. We'll see this more on Thursday.

Remark 1.6. The sheaf category $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S) \subseteq \mathrm{P}(\mathrm{Sm}_S)$ is an ∞-topos, and over a nice base scheme S it is hypercomplete, meaning in particular that it supports a Whitehead's theorem and equivalences can be checked on homotopy groups.

Date: June 23, 2024.

1.2. $\mathbb{A}^1$**-invariance.**

Definition 1.7. We denote by $\mathrm{P}_{\mathbb{A}^1}(\mathrm{Sm}_S) \subseteq \mathrm{P}(\mathrm{Sm}_S)$ the full subcategory of $\mathbb{A}^1$-*invariant* presheaves. That is, exactly those presheaves F for which the projection $\pi\colon X \times \mathbb{A}^1 \to X$ induces an equivalence

$$\pi^*\colon F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$$

for every $X \in \mathrm{Sm}_S$.

Example 1.8. *(Not every representable is* $\mathbb{A}^1$*-invariant)* Representable presheaves need not be $\mathbb{A}^1$-invariant. For example:

- $\mathbb{G}_m$ is $\mathbb{A}^1$ invariant assuming the base S is reduced. This is because it represents units, which are $\mathbb{A}^1$-invariant.
- $\mathbb{A}^1$ is not $\mathbb{A}^1$-invariant, since it represents global sections.

Proposition 1.9. The inclusion $\mathrm{P}_{\mathbb{A}^1}(\mathrm{Sm}_k) \subseteq \mathrm{P}(\mathrm{Sm}_k)$ is a reflective subcategory, and hence admits a left adjoint, which we denote by

$$L_{\mathbb{A}^1}\colon \mathrm{P}(\mathrm{Sm}_k) \to \mathrm{P}_{\mathbb{A}^1}(\mathrm{Sm}_k).$$

Proof. A presheaf F is $\mathbb{A}^1$-invariant if and only if the projection $X \times \mathbb{A}^1 \to X$ induces an equivalence

$$F(X) \xrightarrow{\sim} F(X \times \mathbb{A}^1)$$

for every $X \in \mathrm{Sm}_k$. Via the ∞-categorical Yoneda lemma, this is equivalent to saying that

$$\mathrm{Map}(h_X, F) \xrightarrow{\sim} \mathrm{Map}(h_{X \times \mathbb{A}^1}, F)$$

is an equivalence. The collection of maps $\left\{X \times \mathbb{A}^1 \to X\right\}_{X \in \mathrm{Sm}_k}$, under the Yoneda embedding, forms a class $S \subseteq P(\mathrm{Sm}_k)$, and it is clear to see that a presheaf is $\mathbb{A}^1$-invariant if and only it is S-local. It now suffices to see that the property of being S-local can be checked on a *set* of morphisms. This follows by Sm_k admitting a small skeleton. □

Notation 1.10. We denote by Δ^n the *algebraic n-simplex*

$$\Delta^n := \mathrm{Spec}\left(\mathbb{Z}[t_0, \ldots, t_n]/(\sum t_i - 1)\right).$$

These give a cosimpicial scheme $\Delta^\bullet \in \mathrm{Fun}(\Delta, \mathrm{Sch})$.

Definition 1.11. We define the *singular chains* construction

$$\mathrm{Sing}\colon \mathrm{P}(\mathrm{Sm}_S) \to \mathrm{P}(\mathrm{Sm}_S)$$

by the formula

$$(\mathrm{Sing}F)(X) = \mathrm{colim}_{\Delta^{\mathrm{op}}} F(X \times \Delta^n)$$

Proposition 1.12. We have that $\mathrm{Sing}(F)$ is $\mathbb{A}^1$-invariant for any F.

Proof idea. We want to prove for any $X \in \mathrm{Sch}_S$ that the projection map $\pi\colon X \times \mathbb{A}^1 \to X$ induces an equivalence

$$\pi^*\colon (\mathrm{Sing}F)(X \times \mathbb{A}^1) \to (\mathrm{Sing}F)(X).$$

Let $z\colon X \to X \times \mathbb{A}^1$ denote the zero section, so we'd like to exhibit a simplicial homotopy $\mathrm{id} \simeq z^*\pi^*$. We first show that $\mathrm{id} \simeq z\pi$ as maps of cosimplicial varieties

$$X \times \mathbb{A}^1 \times \Delta^\bullet \to X \times \mathbb{A}^1 \times \Delta^\bullet.$$

We then see a presheaf F preserves simplicial homotopies (any functor does), as does geometric realization. □

Proposition 1.13. *(Singular construction)* The functor $L_{\mathbb{A}^1}$ can be identified with Sing.

Proof idea. Show both constructions are adjoint to the inclusion $\mathrm{P}_{\mathbb{A}^1}(\mathrm{Sm}_S) \subseteq \mathrm{P}(\mathrm{Sm}_S)$. $\square$

In particular this implies that $L_{\mathbb{A}^1}$ preserves finite products, since it is a sifted colimit of right adjoints.

2. Motivic spaces and motivic localization

Definition 2.1. We define the category of *motivic spaces* $\mathrm{Spc}(k)$ as the intersection

$$\mathrm{Spc}(k) = \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) \cap \mathrm{P}_{\mathbb{A}^1}(\mathrm{Sm}_k) \subseteq \mathrm{P}(\mathrm{Sm}_k),$$

that is, the full subcategory of presheaves which are both Nisnevich sheaves and are $\mathbb{A}^1$-invariant.

Problem: Nisnevich sheafifying an $\mathbb{A}^1$-invariant presheaf needs not preserve $\mathbb{A}^1$-invariance, and $\mathbb{A}^1$-localizing a sheaf may break the sheaf condition. There's an explicit example in [MV99, p. 2.7].

To that end, we define *motivic localization* as the infinite composition of both functors.

Definition 2.2. We define $L_{\mathrm{mot}} \colon \mathrm{P}(\mathrm{Sm}_k) \to \mathrm{Spc}(k)$ by the formula

$$L_{\mathrm{mot}} := \mathrm{colim}_{n\to\infty}(L_{\mathbb{A}^1} \circ L_{\mathrm{Nis}})^{\circ n}.$$

Proposition 2.3. We have that L_{mot} preserves finite products. [Hoy14, p. 3.6]

2.1. Some motivic spaces.

For $X \in \mathrm{Sm}_k$ we denote by abuse of notation $X \in \mathrm{Spc}(k)$ to mean the motivic localization of the representable functor associated to X. Given a simplicial set S, we also use S to denote the constant presheaf at S, considered as a motivic space.

We denote by $\mathrm{Spc}(k)_*$ the pointed category of motivic spaces, given as the slice category under the terminal object $\mathrm{Spc}(k)_{*/}$.

We have that $\mathrm{Spc}(k)_*$ is symmetric monoidal under the smash product, defined by

$$X \wedge Y := \frac{X \times Y}{X \vee Y}.$$

Since motivic spaces satisfy descent along Zariski covers, any colimits in Sm_k will become colimits in $\mathrm{Spc}(k)$. In particular we have the following pushout diagram:

$$\begin{array}{ccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}^1 \\ \downarrow & \ulcorner & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \mathbb{P}^1. \end{array}$$

Since each copy of $\mathbb{A}^1$ is contractible, we conclude that

$$\mathbb{P}^1 \simeq \Sigma\mathbb{G}_m = S^1 \wedge \mathbb{G}_m.$$

So we have two flavors of sphere here — one coming from algebraic geometry (e.g. $\mathbb{G}_m$) and one coming from topology (S^1). To that end we introduce bigraded notation for spheres:

$$S^{2,1} = \mathbb{P}^1$$
$$S^{1,1} = \mathbb{G}_m$$
$$S^{1,0} = S^1.$$

Exercise 2.4.

(1) Show the following diagram is a pushout

$$\begin{array}{ccc} X \times Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & \Sigma(X \wedge Y) \end{array}$$

(2) Argue that $\mathbb{A}^n - 0 \simeq S^{2n-1,n}$.

Note that this implies that

$$\frac{\mathbb{A}^n}{\mathbb{A}^n - 0} = \Sigma(\mathbb{A}^n - 0) = S^{2n,n}.$$

2.2. **Functoriality.** The goal here is to argue that the construction $\mathrm{Spc}(S)$ is natural in S in some ways. We follow the exposition in [Hoy17, §4].

Given $f\colon T \to S$, we obtain an induced pullback map

$$\begin{aligned} \mathrm{Sm}_S &\to \mathrm{Sm}_T \\ X &\mapsto X \times_S T, \end{aligned}$$

and this extends to a map of presheaves:

$$f_*\colon \mathrm{P}(\mathrm{Sm}_T) \to \mathrm{P}(\mathrm{Sm}_T),$$

via $(f_* F)(X) = F(X \times_S T)$.

Exercise 2.5. The functor f_* preserves Nisnevich excisiveness and $\mathbb{A}^1$-invariance, hence extends to a functor

$$f_*\colon \mathrm{Spc}(T) \to \mathrm{Spc}(S).$$

Note 2.6. We have that f_* always admits a left adjoint f^*, which if f is smooth is given at the level of presheaves by precomposition with $\mathrm{Sm}_T \to \mathrm{Sm}_S$. In this case we have a further left adjoint $f_\sharp \dashv f^* \dashv f_*$.

Remark 2.7. We have pointed versions of the categories $\mathrm{Spc}_*(S)$, and these functors and adjunctions lift there as well.

2.3. **Thom spaces.** Let $p\colon V \to X$ be an algebraic vector bundle. Then we define $\mathrm{Th}_X(V)$ to be the fiberwise *Thom space* of the bundle. Explicitly, it is a motivic space over X given by

$$\mathrm{Th}_X(V) := \frac{V}{V - X},$$

where $X \hookrightarrow V$ under the zero section.

Example 2.8. The trivial vector bundle over the point $\mathbb{A}^n_k \to \mathrm{Spec}\,(k)$ has Thom space $\mathbb{A}^n/(\mathbb{A}^n - 0)$ which we have seen is the motivic sphere $S^{2n,n}$. More generally, we think about Thom spaces as "twisted sphere bundles."

Given a vector bundle $V \to X$, we denote by

$$\Sigma^V := \mathrm{Th}_X(V) \wedge (-)\colon \mathrm{Spc}_*(X) \to \mathrm{Spc}_*(X)$$

the associated *Thom transformation*.

Proposition 2.9. We will see (probably next lecture) that the Thom transformation Σ^V can be identified with $p_\sharp s_*$, where $s\colon X \to V$ is the zero section.

2.4. **Purity.** One of the key results is *purity*, which essentially says that in motivic spaces, *we can treat every closed immersion as though it is the zero section of a vector bundle.*

Theorem 2.10. (Purity) Let $i\colon Z \hookrightarrow X$ be a closed immersion. Then there is a natural weak equivalence in $\mathrm{Spc}(S)$ of the form

$$\frac{X}{X-Z} \simeq \mathrm{Th}_Z(Ni).$$

sketch. Essentially deformation to the normal cone. We take an $\mathbb{A}^1$-family of closed immersions $Z \times \mathbb{A}^1$ into a space $D_Z X = \mathrm{Bl}_{Z\times 0} X \times \mathbb{A}^1 - \mathrm{Bl}_{Z\times 0} X - 0$ so that at time $t = 1$ it is the normal embedding $Z \hookrightarrow X$ and at $t = 0$ it is the zero section $Z \hookrightarrow N_Z X$. We then show the inclusions

$$\frac{N_Z X}{N_Z X - Z} \to \frac{D_Z X}{D_Z X - Z \times \mathbb{A}^1}$$
$$\frac{X}{X - Z} \to \frac{D_Z X}{D_Z X - Z \times \mathbb{A}^1}$$

are $\mathbb{A}^1$-weak equivalences. □

Remark 2.11. One of the main advantages of purity is that it allows us to identify compactly supported cohomology $\left[\frac{X}{X-Z}, E\right] =: E_Z(X)$ with the ordinary cohomology of Z twisted by the normal bundle, which can be untwisted if the cohomology theory is appropriately oriented. This gives us long exact sequences on Chow groups, algebraic K-theory, etc. where there is a shift in the degree by the codimension of $Z \hookrightarrow X$. What is happening here is a combination of compactly supported cohomology, purity, and orientation data.

References

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