IWOAT SUMMER SCHOOL ON STABLE MOTIVIC HOMOTOPY THEORY, SPRING 2024

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1. Talk 1: Quasicategories

Abstract. Definition of infinity-categories as quasi-categories, the existence of mapping spaces, interpretation. Presentable infinity categories. Bousfield localization. All of this will have to be presented very tersely, mostly as a reminder.

Goal: Set up the basics of infinity categories

1.1. **Quasicategories.** Let $\mathbb{\Delta}$ denote the category whose objects are finite totally ordered sets $[n] := \{0 < 1 < \cdots < n\}$, and whose morphisms are order-preserving functions $[n] \to [m]$.

Definition 1.1. If $\mathscr{C}$ is a category, a *simplicial object* in $\mathscr{C}$ is a functor $\mathbb{\Delta}^{\text{op}} \to \mathscr{C}$, and a *cosimplicial object* is a functor $\mathbb{\Delta} \to \mathscr{C}$.

Simplicial varieties appeared in Grothendieck's work as a way to keep track of covers and descent data, and later appeared in Deligne's work as a combinatorial way to encode the process of resolving singularities. Given a group or group object G, the multiplication and identity maps give rise to a natural cosimplicial object

$$G \to G \times G \leftrightarrows \cdots$$

called the *bar construction*.

Definition 1.2. The category of *simplicial sets* is the functor category $\text{sSet} := \text{Fun}(\mathbb{\Delta}^{\text{op}}, \text{Set})$.

Given a simplicial set

$$\begin{aligned} \mathbb{\Delta}^{\text{op}} &\to \text{Set} \\ [n] &\mapsto X_n, \end{aligned}$$

we call X_n the set of *n-simplices*.

Example 1.3. Any set Y gives rise to a constant simplicial set $\underline{Y}$, given by sending $[n] \mapsto Y$, and every morphism in $\mathbb{\Delta}$ to the identity on Y.

Example 1.4. We denote by Δ^n the simplicial set

$$\Delta^n := \text{Hom}_{\mathbb{\Delta}}(-, [n]) \colon \mathbb{\Delta}^{\text{op}} \to \text{Set}.$$

By the Yoneda lemma, we have a natural bijection

$$\text{Hom}_{\text{sSet}}(\Delta^n, X_\bullet) \cong X_n$$

for any $X_\bullet \in \text{sSet}$.

Example 1.5. If $\mathscr{C}$ is a small 1-category, it gives rise to a simplicial set $N_\bullet \mathscr{C}$, called the *nerve* of $\mathscr{C}$, with the following data:

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- 0-simplices = objects of $\mathscr{C}$
- 1-simplices = morphisms in $\mathscr{C}$
- 2-simplices = pairs of composable morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in $\mathscr{C}$

$\vdots$

- n-simplices = strings of n-composable morphisms

Here the degeneracy maps $(N\mathscr{C}) \to (N\mathscr{C})_{n+1}$ insert an identity, while the face maps $(N\mathscr{C})_n \to (N\mathscr{C})_{n-1}$ compose maps.

In $\mathscr{C}$, composition happens *strictly*, by which we mean there is no notion of homotopy between maps — if $x \xrightarrow{f} y \xrightarrow{g} z$ is a composite of maps, and $h\colon x \to z$, then either $h = g \circ f$, or it is not equal, and this is encoded by the data of a 2-cell:

$$\begin{array}{ccccc} & & y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ x & & \xrightarrow[h]{\qquad\qquad} & & z. \end{array}$$

We think about this 2-cell as a *witness* for the composition.

Q: Given a simplicial set, when can you tell whether it arose as the nerve of a 1-category?

A: Given any diagram of the form $\bullet \to \bullet \to \bullet$, it has to fill in uniquely to a 2-cell. But we also need to fill in composites of three morphisms uniquely (draw a tetrahedron). To that end, let Λ_n^k be the simplicial set obtained from Δ^n by deleting the kth face. This is called a *horn*.

Proposition 1.6. A simplicial set $X_\bullet$ is the nerve of a 1-category if and only if it admits *unique inner horn filling*, meaning for any $0 < k < n$, there is a unique lift:

$$\begin{array}{ccc} \Lambda_n^k & \longrightarrow & X_\bullet \\ \downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

Example 1.7. Let X be a topological space. Then it gives rise to a simplicial set called its *fundamental ∞-groupoid* $\Pi_\infty X$, with the data

- 0-simplices = points $x \in X$
- 1-simplices = paths x to y in X
- 2-simplices = homotopies between paths
- 3-simplices = homotopies between homotopies between paths

$\vdots$

Note that a 2-cell is no longer unique! There can be many homotopies between paths. In particular composition of paths isn't well-defined, in the sense that many paths can function naturally as a composite. We might define $g \circ f$ to be any path together with a 2-cell making the diagram commute:

$$\begin{array}{ccccc} & & y & & \\ & \overset{f}{\nearrow} & & \overset{g}{\searrow} & \\ x & & \dashrightarrow & & z. \end{array}$$

In order to specify a composite now, we need to give the data not only of the 1-cell but also of the 2-cell! This is the vibe of higher-categorical composition. Note that horns don't fill uniquely here.

Exercise 1.8. If you're familiar with the singular chains construction

$$|-|\colon \mathrm{sSet} \leftrightarrows \mathrm{Top}\colon \mathrm{Sing}(-),$$

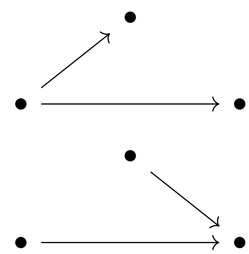
convince yourself that $\mathrm{Sing}(-)$ is the same as $\Pi_\infty(-)$.

Definition 1.9. A *quasicategory* is any simplicial set with (non-unique!) inner horn filling.

Definition 1.10. A *Kan complex* is a quasi-category which also has *outer horn filling*, meaning we have a lift

$$\begin{array}{ccc} \Lambda^k_n & \longrightarrow & X_\bullet \\ \big\downarrow & \nearrow & \\ \Delta^n & & \end{array}$$

not only for $0 < k < n$, but also for $k = 0, k = n$. For $n = 2$, this means we are also allowed to fill the horns:



Exercise 1.11. Show a Kan complex is the nerve of a 1-groupoid if and only if its horn filling is unique.

The functor

$$\begin{aligned} \mathrm{Top} &\to \mathrm{Kan} \\ X &\mapsto \Pi_\infty(X) \end{aligned}$$

is an equivalence of ∞-categories (this is the homotopy hypothesis), hence we can think about spaces as Kan complexes without much loss of generality. We use $\mathcal{S}$ to denote the ∞-category of spaces.

1.2. **Mapping spaces.** We want to make precise the model of quasi-categories as $(\infty, 1)$-categories. The vibe of higher categories is that homs in 1-categories are 0-categories (sets). Homs in 2-categories are 1-categories, homs in 3-categories are 2-categories, etc. Hence homs in $(\infty, 1)$-categories should be $(\infty, 0)$-categories. From the models we're working in:

$$\begin{aligned} (\infty,1)\text{-categories} &= \text{quasi-categories} \\ (\infty,0)\text{-categories} &= \text{Kan complexes,} \end{aligned}$$

we want to argue that, for any quasicategory $\mathscr{C}$, and any pair of objects (0-simplices) $x, y \in \mathscr{C}$, there is a mapping space $\mathrm{Map}_{\mathscr{C}}(x, y)$ which is a Kan complex.

Definition 1.12. For $x, y \in \mathscr{C}$, where $\mathscr{C}$ is a quasicategory, we denote by $\mathrm{Map}_{\mathscr{C}}(x, y)$ the pullback in simplicial sets:

$$\begin{array}{ccc} \mathrm{Map}_{\mathscr{C}}(x,y) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathscr{C}) \\ \big\downarrow & \lrcorner & \big\downarrow {\scriptstyle (\mathrm{ev}_0, \mathrm{ev}_1)} \\ \{x, y\} & \longrightarrow & \mathscr{C} \times \mathscr{C}. \end{array}$$

Here $\mathrm{Fun}(\Delta^1, \mathscr{C})$ denotes an internal hom from the interval Δ^1 to $\mathscr{C}$. The rightmost vertical map is what's called a *bifibration* (the proof that this map is a bifibration is [Lur09, p. 2.4.7.11]), which in particular means that $\mathrm{Map}_{\mathscr{C}}(x, y)$ is a Kan complex.

Remark 1.13. Alternatively we may define $\mathrm{Map}_{\mathscr{C}}(x, y)$ as the simplicial set whose n-simplices are given by the set of all

$$z \colon \Delta^{n+1} \to \mathscr{C},$$

with the $\{n+1\}$-vertex mapping to y, and the vertices $\{0, \dots, n\}$ mapping to x. Technically speaking this is the space of *right morphisms* but when $\mathscr{C}$ is an ∞-category this models the mapping space (it is canonically isomorphic in the homotopy category). As an exercise, verify that $\mathrm{Map}_{\mathscr{C}}(x, y)$ is indeed a Kan complex from the definition.

1.3. **Presentable ∞-categories.** Modulo some set-theoretic technicalities, we can now be content with the existence of a model for infinity-categories. All notions of functors, colimits, adjunctions, etc. should now be understood in the higher categorical sense, i.e. up to higher coherence.

Definition 1.14. [Lur09, p. 5.4.2.1] An ∞-category is *accessible* if it is generated under κ-filtered colimits by a small category.

Example 1.15. The category $\mathcal{S}$ of spaces is presentable, since it admits all colimits and every space is built out of finite CW complexes.

Remark 1.16. By [Lur09, p. 5.4.3.6], a small ∞-cat is accessible if and only if it is idempotent complete.[1] so finitely generated free R-modules fail to contain retracts (projectives) so they're not idempotent complete and hence not accessible.

Definition 1.17. Given any ∞-category $\mathscr{C}$, we denote by $\mathrm{P}(\mathscr{C}) := \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathcal{S})$ the category of (∞-)*presheaves.*

Example 1.18. By the previous two examples, $\mathrm{P}(\mathscr{C})$ is presentable for any $\mathscr{C}$. This is the coYoneda lemma — that any presheaf is a colimit of representable ones.

Definition 1.19. We say an ∞-category $\mathscr{C}$ is *presentable* if it is accessible and admits all colimits (cocomplete).

Theorem 1.20. *(Adjoint functor theorem)* Let $F \colon \mathscr{C} \to \mathscr{D}$ be a functor between presentable categories. Then

- F admits a right adjoint if and only if F preserves all colimits
- F admits a left adjoint if and only if it preserves all limits and κ-filtered colimits

Really hard to write down functors explicitly in quasi-categories, since we are writing down a map of simplicial sets, which is a lot of data. AFT is nice because it lets us get functors without writing them explicitly, but they are still characterized by being adjoints.

Notation 1.21. We denote by Pr^L the category of presentable ∞-categories and colimit-preserving functors between them. Note every functor in Pr^L is a left adjoint.

[1] Idempotent complete has a number of definitions, in particular it implies that idempotent endomorphisms $f \colon X \to X$ (i.e. $f \circ f = f$) correspond bijectively to retracts of X, i.e. composites $Y \hookrightarrow X \to Y$. If $\mathscr{C}$ is idempotent complete then it is closed under retracts.

1.4. **Localization.**

Definition 1.22. [Lur09, p. 5.2.7.2] A functor $f\colon \mathscr{C} \to \mathscr{D}$ is a *localization* if it admits a fully faithful right adjoint.

In many cases a localization is given by inverting a class of morphisms in $\mathscr{C}$. In particular let $S \subseteq \operatorname{mor}\mathscr{C}$ be a class of morphisms in $\mathscr{C}$, then we can try to *invert* S by cooking up a new category $\mathscr{C}[S^{-1}]$.

Definition 1.23. [Lur09, p. 5.5.4.1] Let $S \subseteq \operatorname{mor}\mathscr{C}$. We say $z \in \mathscr{C}$ is *S-local* if for every $s\colon x \to y$ in S, the induced map

$$\operatorname{Map}_{\mathscr{C}}(y, z) \to \operatorname{Map}_{\mathscr{C}}(x, z)$$

is an equivalence.

Remark 1.24. Let $\mathscr{C}_0 \subseteq \mathscr{C}$ be the full subcategory of S-local objects. If this admits a left adjoint, it makes sense to call that adjoint L_S, that is, S-localization, since it inverts every morphism in S. *This is where presentable categories give us an advantage.* In general arguing for the existence of a left adjoint isn't easy, however if $\mathscr{C}$ is presentable, then the adjoint functor theorem tells us that we just have to check the inclusion $\mathscr{C}_0 \subseteq \mathscr{C}$ preserves limits and filtered colimits.

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Proposition 1.25. [Lur09, p. 5.5.4.15] If $\mathscr{C}$ is presentable and $S \subseteq \operatorname{mor}\mathscr{C}$ is small, then the inclusion of the full subcategory of S-local objects admits a left adjoint.[2]

Example 1.26. In the next talk, our primary application of this machinery will be looking at the presheaf category $\mathrm{P}(\mathscr{C})$, which is presentable by Example 1.18. We can look at full subcategories of presheaves which satisfy a certain sheaf condition and argue this is a reflective subcategory hence we will have an adjoint we call *sheafification*.

Remark 1.27. Given a class of arrows $S \subseteq \operatorname{mor}\mathscr{C}$, we can always form $\mathscr{C}[S^{-1}]$ by adjoining formal inverses to S and considering all composites of morphisms in $\mathscr{C}$ and formal inverses (zig-zags). This is called *Dwyer–Kan localization* or *hammock localization*. This satisfies the correct universal property of localization, but we might encounter size issues. Bousfield localization is a particular example of Dwyer–Kan localization, but where we are able to guarantee that we don't encounter any size issues since the localization is a subcategory of the original category.

References

[Lur09] Jacob Lurie. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. DOI: 10.1515/9781400830558. URL: https://doi.org/10.1515/9781400830558.

[2]The terminology for this is that $\mathscr{C}_0 \subseteq \mathscr{C}$ is a *reflective subcategory*.