

BITANGENTS TO SYMMETRIC QUARTICS

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ABSTRACT. Recall that a non-singular planar quartic is a canonically embedded non-hyperelliptic curve of genus three. We say such a curve is *symmetric* if it admits non-trivial automorphisms. The classification of (necessarily finite) groups appearing as automorphism groups of non-singular curves of genus three dates back to the last decade of the 19th century. As these groups act on the quartic via projective linear transformations, they induce symmetries on the 28 bitangents. Given such an automorphism group $G = \text{Aut}(C)$, we leverage tools from equivariant homotopy theory to prove that the G -orbits of the bitangents are independent of the choice of C , and we compute them for all twelve types of smooth symmetric planar quartic curves. We further observe that techniques deriving from equivariant homotopy theory directly reveal patterns which are not obvious from a classical moduli perspective.

1. INTRODUCTION

The study of bitangents to plane quartics dates back to the 19th century, and has been one of the pillars of classical enumerative algebraic geometry. In 1850, Jacobi [Jac50] was the first to compute that the number of bitangents to a smooth planar quartic was 28, using work of Plücker from two decades earlier [Plü34]. This was extended to general quartics shortly thereafter by Hesse [Hes55]. The configurations of these lines, their combinatorics, and their sweeping connections throughout mathematics were objects of deep fascination throughout the 19th century, and their connections to other fields of geometry, e.g., Lie algebras, continue to be topics of interest to modern mathematicians [Man06]. The automorphism groups of genus three (non-hyperelliptic) curves have been completely classified, and we provide a historical overview of this work in Subsection 2.2. Since a planar quartic is a canonically embedded non-hyperelliptic genus three curve, any automorphism of a genus three curve will extend to an automorphism of the projective plane, and such an automorphism group will be finite by the Hurwitz formula. In particular the bitangents of such a curve will inherit a natural action from the automorphism group of the curve. Leveraging tools from equivariant homotopy theory, we prove the action on the bitangents is independent of the choice of symmetric quartic, and we compute it in each case.

Theorem 1.1. Let G be any group which appears as the automorphism group of a genus three non-hyperelliptic curve. Then for any such curve C with $G \cong \text{Aut}(C)$, the action of G on the bitangents of C is independent of the choice of C , and is given by the following G -set:¹

¹See Section 4 for an explanation of the notation.

Type	Aut(C)	Order	28 bitangents
I	$\mathrm{PSL}_2(7)$	168	$[\mathrm{PSL}_2(7)/S_3]$
II	$C_4^{\times 2} \rtimes S_3$	96	$[C_4^{\times 2} \rtimes S_3/C_6] + [C_4^{\times 2} \rtimes S_3/C_8]$
III	$C_4 \odot A_4$	48	$[C_4 \odot A_4/C_2] + [C_4 \odot A_4/C_{12}]$
IV	S_4	24	$[S_4/C_2^o] + [S_4/C_2^e] + [S_4/S_3]$
V	P	16	$[P/C_2^{(1)}] + [P/C_2^{(2)}] + [P/C_2^{(3)}] + [P/C_4^Z]$
VI	C_9	9	$3[C_9/e] + [C_9/C_9]$
VII	D_8	8	$[D_8/e] + 2[D_8/C_2^{(1)}] + 2[D_8/C_2^{(2)}] + [D_8/C_2^Z]$
VIII	C_6	6	$4[C_6/e] + [C_6/C_2] + [C_6/C_6]$
IX	S_3	6	$3[S_3/e] + 3[S_3/C_2] + [S_3/S_3]$
X	K_4	4	$4[K_4/e] + 2[K_4/C_2^L] + 2[K_4/C_2^\Delta] + 2[K_4/C_2^R]$
XI	C_3	3	$9[C_3/e] + [C_3/C_3]$
XII	C_2	2	$12[C_2/e] + 4[C_2/C_2]$

Example 1.2. As an example, we visualize the 28 bitangents on the following *Edge quartic*.² This admits a clear D_8 symmetry on the xy -plane, which extends to an S_4 -symmetry by passing between the different affine patches.

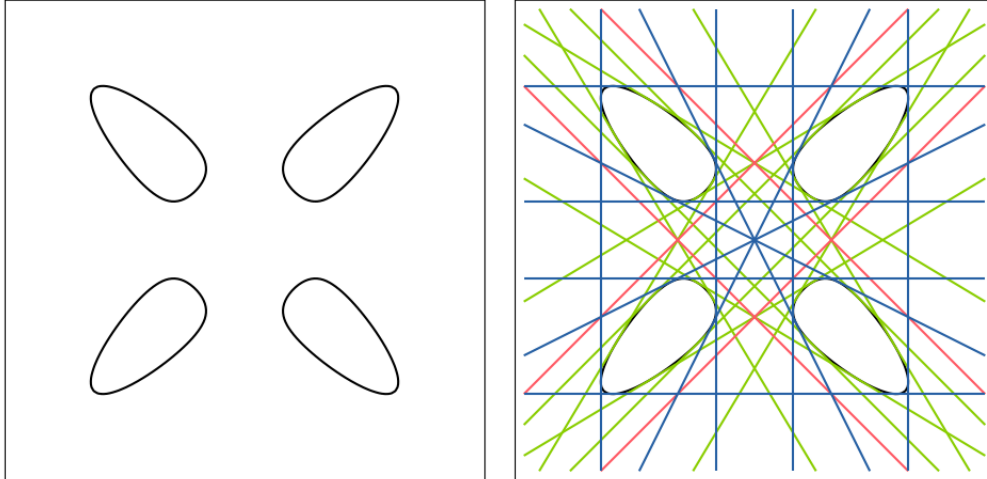


FIGURE 1. The *Edge quartic*, defined by the equation $25(x^4 + y^4 + z^4) - 34(x^2y^2 + x^2z^2 + y^2z^2)$, and its 28 real bitangents graphed with their S_4 orbits.

We prove Theorem 1.1 by leveraging the machinery of [Bra24] by first formulating the count of bitangents on a G -symmetric quartic as a G -equivariant vector bundle, and computing its MU_G -valued Euler class — this proves that the action of $G \cong \mathrm{Aut}(C)$ on the bitangents is independent of C . We pick candidate curves in each automorphism class from [Dol12, Hen76] and compute equations for each of the bitangents using code modified from [PSV11]. After this we determine the orbits of lines and their isotropy under the action of the automorphism group. We provide figures in each case, and discuss rationality where it is relevant.

²See Remark 4.4 for a further discussion of Edge quartics.

Remark 1.3. There are multiple ways to argue that the action of the automorphism group on bitangents (and even θ -characteristics) is independent of the choice of curve, as in Theorem 1.1. A classical way is to observe that each component of the moduli space is connected, although we remark that our technology works even when this might fail. After finishing this preprint, the authors were made aware of a third way to prove this independence, which had appeared in a preprint of Braden and Disney–Hogg a few months prior [BDH24] by leveraging that curves in the same component of the moduli space have equivalent rational representations, and the orbit decomposition can be recovered by the rational representation. In particular the table in our Theorem 1.1 can be found in Table 6 of [BDH24], although the authors there do not specify the isotropy type of each orbit.

Remark 1.4. (*Other related literature*)

- The orbits of the automorphism group of the Klein quartic on its 28 bitangents is well-studied, and visualizable over $\mathbb{P}_{\mathbb{F}_7}^1$ ([Cox83, JvOS94]). In general the equivariant enumerative geometry of the Klein quartic has been a popular direction of research since the mid-to-late 1800’s — the action on the even theta characteristics is in [DK93], and the action on the Steiner complexes and Aronhold sets in [JvOS94]. See [Dol12, Remark 6.5.4] for more information.
- The bitangents of non-hyperelliptic curves of genus three with cyclic automorphisms of degree 3, 6, or 9 was recently studied in a paper of Liang [Lia20].

Remark 1.5. (*On rationality*) In 1873, Zeuthen proved that a smooth real quartic can have 3, 8, 16, or 28 bitangents, and this depends precisely on the topology of the real curve [Zeu74]. One might ask whether results in this paper can make general statements about the number of real bitangents to smooth real planar quartics with given automorphism group, in a way analogous to [Bra24, Theorem 1.3]. When the automorphism group is real (lies in $\mathrm{PGL}_3(\mathbb{R})$), orbits of odd size necessarily are defined over the reals. Beyond this, we cannot say much as there isn’t a clear connection between automorphism types (in the decomposition in Theorem 1.1) and the real topology of a curve — we can observe this explicitly by traveling through a family of curves of a given type and seeing that the number of real bitangents can vary wildly. For example a Type XII quartic can have no real points, two non-nested ovals, two nested ovals, or three ovals.

1.1. **Outline.** The paper is organized as follows. Section 2 gives background information on canonical quartic plane curves which groups can arise as non-trivial automorphisms of them. Section 3 defines the vector bundle which parameterizes the enumerative problem of counting bitangents to symmetric quartics, following the construction of [LV21], shows it’s an equivariant vector bundle for any group G which arises as a non-trivial automorphism group of a canonical quartic, and leverages the equivariant Euler number of [Bra24] to conclude that the equivariant count of bitangents on a G -symmetric quartic is independent of the choice of quartic. Section 4 gives example quartics for each group and graphs orbits of real bitangents for each. Section 5 concludes the paper with a subgroup lattice of automorphism groups of canonical quartics.

1.2. **Data availability.** Throughout the paper are various **Computation** environments. These refer to numerical solutions for bitangents to example quartic curves for each type, whose orbits under the automorphism group of the curve were then checked on a computer

(but can also be verified by hand). Explicit equations for all the lines for the example quartics referenced, grouped into their orbits, as well as the code the authors used to compute bitangents to planar quartics is available here:

<https://tbrazel.github.io/supplementary/lines.pdf>

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2. BACKGROUND

2.1. Quartics, the basics. Nonsingular curves of genus $g \geq 2$ come in two flavors, being *canonical* and *hyperelliptic*, depending on whether the canonical class is very ample. When the canonical class of C is very ample, we have that $\Gamma(C, \Omega_C^1)$ is a vector space of dimension g , and a choice of basis of holomorphic forms defines an embedding $C \rightarrow \mathbb{P}^{g-1}$ called a *canonical embedding*. Any automorphism induces an automorphism of Ω_C^1 by pullback, and therefore extends to an automorphism of \mathbb{P}^{g-1} , hence the automorphism groups of canonical curves are all *geometric* in the sense that they act on the embedded curve via projective linear transformations.

It is a classical fact that a general non-hyperelliptic curve of genus $g \geq 3$ admits no nontrivial automorphisms. This motivates the following definition.

Definition 2.1. We say a non-hyperelliptic non-singular curve of genus g is *symmetric* if it admits any automorphism other than the identity.

2.2. Automorphisms of smooth quartics. Here we provide a highly incomplete history of the classification of the automorphism groups of genus three non-singular hyperelliptic curves. This is a history which is scattered throughout the literature, and references papers plagued by minor errors, so we feel it is worth recounting.

In 1878, Schwarz proved that the automorphism group of a compact Riemann surface of genus $g \geq 2$ is finite. One argues this by observing that for a non-hyperelliptic curve, the automorphism group acts faithfully on the Weierstrass points of the curve, of which there are only finitely many. In 1892, Hurwitz determined an upper bound on the size of such automorphism groups, being $84(g-1)$ [Hur92]. This bound is not always attained, however a famous example is the so-called *Klein quartic* (Type I in our classification), which was studied extensively by Felix Klein [Kle78]. When $g = 3$, the Klein quartic has an automorphism group of size 168, attaining the upper bound proven by Hurwitz.

During this time, Kantor had been working on understanding what kinds of periodic transformations existed on the projective plane, informed by algebraic questions arising from the Hermite and Frobenius programs on quadratic forms [Kan95a]. In modern language, we might understand this as the study of cyclic subgroups of $\mathrm{PGL}_3(\mathbb{C})$. In the course of this work, Kantor observed the following result.

Theorem 2.2. [Kan95b, p. 40] Given any finite subgroup $G \leq \mathrm{PGL}_3(\mathbb{C})$, there exists a planar curve of genus $g \geq 3$ which is G -invariant.

Given this result, one might fix a genus g and study which finite groups can act on smooth non-hyperelliptic planar curves of genus g , and one may ask what the full automorphism groups of such curves are. Kantor claimed in this same work to classify the automorphism groups of plane quartic curves, however this classification contained many errors which were correctly shortly thereafter by Wiman in 1896 [Wim96]. In particular, following Hurwitz' work [Hur92], one knows that the prime factors of $\#\mathrm{Aut}(C)$ will only ever be 2, 3, or 7, which was known to Wiman [Wim96, p. 223]. Although it does not satisfy modern standards of rigor, Wiman correctly identifies all the possible orders of automorphism groups of genus three non-hyperelliptic non-singular curves, and provides an indication as to the generators of these groups.

In 1910, Ciani extended this classification to all irreducible quartics (not necessarily nonsingular) [Cia10]. There are subtleties, however, in such a classification, as we can see in the following example, where we drift beyond the reach of the Hurwitz theorem.

Example 2.3. Observe that the quartic $x^3y + z^4$ (which is singular at the point $[0 : 1 : 0]$) admits a symmetry of arbitrary order $r \geq 7$ given by $(x, y, z) \mapsto (\zeta_r x, \zeta_r^{r-3} y, z)$ [Cia10, N. 19]. In particular its automorphism group is necessarily infinite.

Throughout the 20th century, it is unclear which results were known or lost in the literature. Henn recomputed the automorphism groups of genus three non-hyperelliptic curves via the action on Weierstrass points in his 1976 work, obtaining a classification of finite groups occurring as automorphism groups of smooth planar quartics [Hen76]. This essentially recovered the computation done by Wiman nearly a century earlier.

Around the mid-1960's, many mathematicians published work studying compact Riemann surfaces equipped with non-trivial involutions. This body of work is far too encompassing to mention all of the relevant players, but we highlight a particular program of mathematics emanating from work of A. Kuribayashi, which studied the moduli of compact Riemann surfaces equipped with non-trivial automorphisms [Kur66, Kur76], leveraging tools from Teichmüller theory. In [KK77] the authors study smooth projective genus three curves which are cyclic covers of the projective line or of an elliptic curve, and determine their automorphism groups, again recovering the classification initially due to Wiman.

At the dawn of the 21st century, as computer-aided research became increasingly ubiquitous in mathematics, Breuer made tremendous progress by determining all finite groups which act on a curve of genus ≤ 48 [Bre00]. This leveraged group-theoretic constraints on such groups arising from covering space theory, and was accomplished via an exhaustive search through a library of finite groups in GAP [GAP24]. Magaard, Shaska, Shpectorov and Völkein provide a lovely discussion of how to re-derive the classification for genus $g = 3$ from the data obtained by Breuer (that paper also contains rich historical background for the interested reader) [MSSV02].

A beautiful resource for automorphisms of quartics can be found in Dolgachev's book [Dol12, §6.5], which recasts the research of Kantor and Wiman in modern language and rigor. Before this book was published, a fantastic paper of Bars [Bar05] sketches the classification of automorphism groups of quartics, following both Henn's thesis and the aforementioned paper

of Komiya and Kuribayashi. Bars' note corrected errors in the classification appearing in the lecture notes which would later become Dolgachev's book.

With all this in mind, we state the following classification, which we feel it is fair to attribute to Wiman, although we refer the reader to [Dol12, p. 270] for a rigorous proof.

Theorem 2.4. [Wim96] The following groups G are the only groups occurring as the automorphism group of a smooth plane quartic. In each case we provide an equation for quartics $F(x, y, z)$ of that type, with generators for its automorphism group (c.f. [Bar05, Dol12]). Here ζ_n denotes a primitive n th root of unity, and in $\mathrm{PSL}_2(7)$ we have $\alpha := \frac{\zeta_7^5 - \zeta_7^2}{\sqrt{-7}}$, $\beta = \frac{\zeta_7^6 - \zeta_7}{\sqrt{-7}}$ and $\gamma = \frac{\zeta_7^3 - \zeta_7^4}{\sqrt{-7}}$

Type	$\mathrm{Aut}(C)$	GAP ID	$F(x, y, z)$	Generators in $\mathrm{PGL}_3(\mathbb{C})$
I	$\mathrm{PSL}_2(7)$	[168, 42]	$x^3y + y^3z + z^3x$	$\begin{pmatrix} \zeta_7^4 & 0 & 0 \\ 0 & \zeta_7^2 & 0 \\ 0 & 0 & \zeta_7 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & \gamma & \beta \\ \beta & \alpha & \gamma \\ \gamma & \beta & \alpha \end{pmatrix}$
II	$C_4^{\times 2} \rtimes S_3$	[96, 64]	$x^4 + y^4 + z^4$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} -i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & i & 0 \end{pmatrix}$
III	$C_4 \odot A_4$	[48, 33]	$x^4 + y^4 + z^4 + (4\zeta_3 + 2)x^2y^2$	$\begin{pmatrix} \frac{1+i}{2} & \frac{-1+i}{2} & 0 \\ \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}, \begin{pmatrix} \frac{1+i}{2} & \frac{-1-i}{2} & 0 \\ -\frac{1+i}{2} & \frac{-1+i}{2} & 0 \\ 0 & 0 & \zeta_3^2 \end{pmatrix}$
IV	S_4	[24, 12]	$x^4 + y^4 + z^4 + a(x^2y^2 + y^2z^2 + z^2x^2)$	$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
V	P	[16, 13]	$x^4 + y^4 + z^4 + ax^2y^2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}$
VI	C_9	[9, 1]	$x^4 + xy^3 + yz^3$	$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_9 \end{pmatrix}$
VII	D_8	[8, 3]	$x^4 + y^4 + z^4 + ax^2y^2 + bxyz^2$	$\begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
VIII	C_6	[6, 2]	$z^3y + x^4 + ax^2y^2 + y^4$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}$
IX	S_3	[6, 1]	$x^3z + y^3z + x^2y^2 + axyz^2 + bz^4$	$\begin{pmatrix} \zeta_3 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
X	K_4	[4, 2]	$x^4 + y^4 + z^4 + ax^2y^2 + by^2z^2 + cx^2z^2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
XI	C_3	[3, 1]	$z^3y + x(x-y)(x-ay)(x-by)$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \end{pmatrix}$
XII	C_2	[2, 1]	$x^4 + y^4 + z^4 + x^2(ay^2 + byz + cz^2) + dy^2z^2$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The equations are subject to the following constraints:

- Type IV: we have $a \neq 0$ since this yields a Type II curve, and $a \neq \frac{3}{2}(-1 \pm \sqrt{-7})$, otherwise one obtains a Type I curve.
- Type V: $a \notin \{0, \pm 2\sqrt{-3}, \pm 6\}$. If $a = 0$ or $a = \pm 6$ we obtain a Type II quartic, and if $a = \pm 2\sqrt{-3}$ we obtain a Type III quartic.
- Type VII: $b \neq 0$. For $b = 0$ we obtain a Type V quartic.
- Type VIII: $a \neq 0$. For $a = 0$ we obtain a Type III quartic.
- Type IX:³ $a \neq 0$, otherwise we obtain a Type IV quartic.
- Type X: all the values $\{\pm a, \pm b, \pm c\}$ are distinct, otherwise we obtain a Type VII curve.
- Type XI: if either $a = \zeta_3$ or $b = \zeta_3$ we obtain a Type VI curve. If either $a = -1$ or $b = -1$ we obtain a Type VIII curve.
- Type XII:⁴ If $b = 0$ we obtain a Type X quartic. If $a = -2$, $b = 0$, and $c = -d$, we obtain a Type IX curve.

The reader may notice subgroup containment between some of these groups — we discuss how this manifests in our computations in Section 5.

Remark 2.5. The quartic equations appearing in Theorem 2.4 are in fact the only quartic equations (up to projective transformation) with the given automorphism group. One proves this by first providing normal forms for cyclic actions, then doing a case-by-case analysis as in [Dol12, §6.4]. When free parameters appear, they correspond to the dimension of the moduli of smooth planar quartics having the given automorphism group. In that sense, the free coefficients in the equations Theorem 2.4 give an indication of how the moduli of symmetric quartics is stratified.

3. THE EULER NUMBER FOR BITANGENTS

The proof of Theorem 1.1 relies on the fact that we can obtain a G -equivariant count of the 28 bitangents to a smooth quartic C for $G \cong \text{Aut}(C)$ using the equivariant Euler number of [Bra24], and this count will be independent of choice of quartic with automorphism group G . We use the bundle parameterizing bitangents to smooth quartics following the construction of [LV21], which we recount exactly below.

Let

$$\mathcal{S} \rightarrow (\mathbb{P}^2)^\vee$$

denote the tautological bundle over the moduli of lines in \mathbb{P}^2 . Let $X := \mathbb{P}(\text{Sym}^2 \mathcal{S}^\vee)$ so that a point on X is a pair (L, Z) where $L \subseteq \mathbb{P}^2$ is a line and $Z \subseteq L$ is a degree two subscheme.

³Our equation differs slightly from [Dol12] by rescaling variables.

⁴Our equation for a Type XII quartic matches that found in [Dol12, Theorem 6.5.2], which is a priori different from [Dol12, Lemma 6.5.1] and [Hen76, Bar05]. A Type XII quartic is of the form $x^4 + x^2 L_2(y, z) + L_4(y, z)$, where L_k denotes a homogeneous binary form of degree k . By the classical theory of quartics, we recall that every binary quartic form admits a *canonical form* $y^4 + dy^2 z^2 + z^4$ after a projective linear transformation, a fact that dates back to Cayley but can be found in [Ell13, Chapter XII]. This is the source of the quartic equation we use here.

Writing $\pi: X \rightarrow \mathbb{P}^{2\vee}$ for the natural projection map, we have the commutative diagram:

$$(1) \quad \begin{array}{ccc} \pi^*\mathrm{Sym}^2\mathcal{S}^\vee & \longrightarrow & \mathrm{Sym}^2\mathcal{S}^\vee \\ \downarrow & & \downarrow \\ X & \xrightarrow{\pi} & (\mathbb{P}^2)^\vee. \end{array}$$

Note that any fiber of $\mathcal{O}_X(-1) \subseteq \pi^*\mathrm{Sym}^2\mathcal{S}^\vee$ determines a choice of equations defining $Z \subseteq L$, whence the quotient of

$$(2) \quad \mathcal{O}_X(-1) \rightarrow \pi^*\mathrm{Sym}^2\mathcal{S}^\vee$$

parameterizes quadratic equations on L vanishing along Z . Squaring, we obtain the composite

$$(3) \quad \mathcal{O}_X(-2) \rightarrow \mathrm{Sym}^2\mathcal{S}^\vee \otimes \mathrm{Sym}^2\mathcal{S}^\vee \rightarrow \mathrm{Sym}^4\mathcal{S}^\vee.$$

The vector bundle for our enumerative problem will be the quotient bundle

$$E := \mathrm{Sym}^4\mathcal{S}^\vee / \mathcal{O}_X(-2),$$

which will parameterize quartic polynomials on L vanishing at the square of an equation of Z . In other words, the quotient $\mathrm{Sym}^4\mathcal{S}^\vee \rightarrow E$ over the fiber (L, Z) in

$$(4) \quad 0 \rightarrow \mathcal{O}_X(-2) \rightarrow \mathrm{Sym}^4\mathcal{S}^\vee \rightarrow E \rightarrow 0$$

realizes E as the evaluation of a quartic polynomial on L along $2Z$.

Theorem 3.5. Let $G \leq \mathrm{PGL}_3$ be contained in any group appearing as the automorphism group of a smooth non-hyperelliptic plane quartic. Then

$$E \rightarrow X$$

constructed above is a G -equivariant vector bundle, and any planar quartic C with $G \leq \mathrm{Aut}(C)$ induces a G -equivariant section of this bundle whose zeros count the bitangents to C .

Proof. The G -action on \mathbb{P}^2 defines an action on $\mathbb{P}^{2\vee}$ and thus on $\mathcal{S} \rightarrow (\mathbb{P}^2)^\vee$. The symmetric power $\mathrm{Sym}^2\mathcal{S}^\vee \rightarrow (\mathbb{P}^2)^\vee$ is thus a G -vector bundle, and the projectivization $X := \mathbb{P}(\mathrm{Sym}^2\mathcal{S}^\vee) \xrightarrow{\pi} (\mathbb{P}^2)^\vee$ is as well. As the actions on X , $\mathrm{Sym}^2\mathcal{S}^\vee$, and $\pi^*\mathrm{Sym}^2\mathcal{S}^\vee$ are all inherited from that on $(\mathbb{P}^2)^\vee$, diagram (1) commutes equivariantly.

Since the G -action on $\mathcal{O}_X(-1)$ is inherited from that of $\pi^*\mathrm{Sym}^2\mathcal{S}^\vee$, the inclusion $\mathcal{O}_X(-1) \rightarrow \pi^*\mathrm{Sym}^2\mathcal{S}^\vee$ is G -equivariant. Squaring, we deduce that

$$\mathcal{O}_X(-2) \hookrightarrow (\pi^*\mathrm{Sym}^2\mathcal{S}^\vee)^{\otimes 2}$$

is G -equivariant as well. Since over any particular (L, Z) in X , the composition

$$\mathcal{O}_X(-2)_{(L,Z)} \hookrightarrow ((\mathrm{Sym}^2\mathcal{S}^\vee)^{\otimes 2})_{(L,Z)} \rightarrow (\mathrm{Sym}^4\mathcal{S}^\vee)_{(L,Z)}$$

is just the composition of inclusions of the space spanned by the square of a quadratic on Z , $\langle q_Z^2 \rangle \mapsto \langle q_Z^2 \rangle \mapsto \langle q_Z^2 \rangle$, into the space $(\mathrm{Sym}^4\mathcal{S}^\vee)_{(L,Z)}$ of all degree 4 polynomials on L ,

$$\mathcal{O}_X(-2) \rightarrow \mathrm{Sym}^4\mathcal{S}^\vee$$

is G -equivariant. In order to argue that the quotient bundle E can be chosen equivariantly, we pick a G -invariant Hermitian metric in $\mathrm{Sym}^4 \mathcal{S}^\vee$ in order to induce an equivariant splitting of the short exact sequence of bundles

$$\mathcal{O}_X(-2) \rightarrow \mathrm{Sym}^4 \mathcal{S}^\vee \rightarrow E.$$

Therefore $E \rightarrow X$ is a G -equivariant vector bundle, and G -equivariant planar quartics induce equivariant sections of the bundle. \square

We state this conclusion again explicitly:

Corollary 3.6. Let G be as in Theorem 2.4. Then the bitangents of any smooth planar quartic which is G -symmetric have G -orbits given by the Euler class $n_G(E)$.

If f is a quartic polynomial defining a G -symmetric, smooth, non-hyperelliptic plane quartic C and σ_f is the section $X \rightarrow E$ determined by C , then σ_f vanishing at (L, Z) exactly means that L is a bitangent of C with points of bitangency at $Z \subset L$. Thus the fact that σ_f is a G -equivariant section and C has 28 bitangents exactly means that $Z(\sigma_f) = \{(L, Z) : \sigma_f(L, Z) = 0\}$ is a size 28 G -set, and G acts transitively on the disjoint union of orbits of bitangents and points of bitangency of C . This count of bitangents is independent of symmetric quartic C with $\mathrm{Aut}(C) \cong G$ since the Euler number is independent of choice of section:

Corollary 3.7. Let G be any group which appears as the automorphism group of a genus three non-hyperelliptic curve. Then for any such curve C with $G \cong \mathrm{Aut}(C)$, the action of G on the bitangents of C is independent of the choice of C .

Proof. This is a direct consequence of equivariant conservation of number, pulling back an Euler class for equivariant homotopical bordism to the Burnside ring of the group [Bra24, Theorem 5.24]. \square

Combining Theorem 3.5 and Corollaries 3.6 and 3.7 proves Theorem 1.1. Formulas in $A(G)$ for the 28 bitangents of G -symmetric quartics are given in the table included with the statement of the main theorem in the introduction (see Theorem 1.1), which we omit here for brevity.

Remark 3.8. It is worth noting that for any subgroup H of G we have a restriction map $\mathrm{Res}_H^G: A(G) \rightarrow A(H)$ that sends the class of any G -set to the H -set with H -action induced from the action of G . In particular, any G -symmetric quartic C is H -symmetric, and a G -action on the bitangents of C descends to an H -action on the bitangents of C via restriction. We explore consequences of this for equivariant counts of bitangents in Section 5.

4. SYMMETRIC QUARTICS AND THEIR BITANGENTS

In this section we compute the action of the automorphism group of a symmetric quartic on its 28 bitangents for all 12 types. For $H \leq G$ a subgroup, we denote by $[G/H]$ the isomorphism class of the G -set of left cosets of H in G . Every genuine, finite G -set decomposes as such a sum.

4.1. **Type I: Klein quartics.** The most well-studied symmetric quartic, which boasts its own book [Lev01], and its own sculpture at SLMath (formerly MSRI)⁵, is the so-called *Klein quartic*:

$$x^3y + y^3z + z^3x.$$

Its automorphism group G_{168} is the maximum possible by the Hurwitz bound, making it a prototypical example of what is called a *Hurwitz curve*. Its automorphism group is isomorphic to the projective special linear group over the finite field with seven elements, which acts on \mathbb{P}^2 in the following way:

$$\mathrm{PSL}_2(7) \cong \left\langle \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \left(\begin{array}{ccc} -i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & i & 0 \end{array} \right) \right\rangle \leq \mathrm{PGL}_3(\mathbb{C}).$$

We give a partial indication as to how to visualize this group. The Klein quartic, considered as a Riemann surface, can be tiled with 24 heptagons, and in fact it is a quotient of the hyperbolic plane via the congruence subgroup $\Gamma(7) \leq \mathrm{PSL}(2, \mathbb{Z})$, whose quotient $\mathrm{PSL}_2(7)$ acts via deck transformations and becomes the automorphism group of the Klein quartic. This group can be understood as the automorphisms attached to the tiling — a heptagon can be sent to any other of the 24 heptagons, and has a sevenfold rotation, for a grand total of $24 \times 7 = 168 = \#\mathrm{PSL}_2(7)$ symmetries.⁶

Alternatively via the automorphism $\mathrm{PSL}_2(7) \cong \mathrm{PSL}_3(2)$, the automorphism group of the Klein quartic can be visualized as the symmetries of the *Fano plane* as in Figure 2.

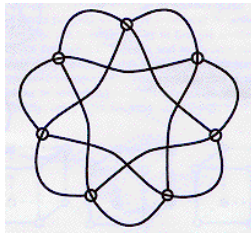


FIGURE 2. An artistic rendering of the sevenfold symmetries on the Fano plane, drawn by Burkard Polster, found on John Baez' blog [Bae13].

The automorphism group acts on bitangents, so we recover our first computation, which was undoubtedly known to Klein.

Computation 4.1. The automorphism group of the Klein quartic acts transitively on the 28 bitangents, with isotropy given by the symmetric group $S_3 \leq \mathrm{PSL}_2(7)$.

The Klein has exactly four real bitangents, which we visualize in Figure 3.

⁵The sculpture is called *The Eightfold Way*, created by Helaman Ferguson and installed in 1993. Thurston wrote a paper about the Klein quartic in honor of the statue's inauguration [Thu99].

⁶Reflections of the heptagons are not considered since they don't preserve the complex structure.

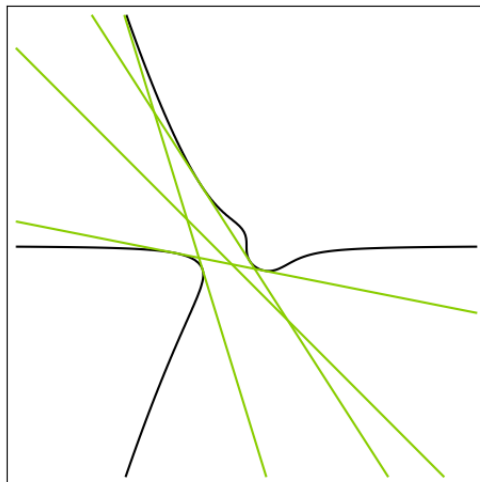


FIGURE 3. The four real bitangents graphed on the Klein quartic. All lines lie in the same orbit.

4.2. **Type II: the Dyck quartic.** The quartic of type two has equation

$$x^4 + y^4 + z^4,$$

and was first studied in an 1880 paper by W. Dyck [Dyc80], who ascertained that it admits 96 symmetries. The automorphism group has isomorphism type $G_{96} \cong C_4^{\times 2} \rtimes S_3$, acting on the projective plane in the following way:

$$(2) \quad C_4^{\times 2} \rtimes S_3 \cong \left\langle \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \left(\begin{array}{ccc} -i & 0 & 0 \\ 0 & 0 & 1 \\ 0 & i & 0 \end{array} \right) \right\rangle \leq \mathrm{PGL}_3(\mathbb{C}).$$

Computation 4.3. Any Type II quartic has bitangents with orbits

$$[C_4^{\times 2} \rtimes S_3/C_6] + [C_4^{\times 2} \rtimes S_3/C_8].$$

While the Dyck quartic has no real points, it does admit real bitangents, where the points of tangency are complex. These are given by

$$\begin{aligned} x - y - z \\ x + y - z \\ x - y + z \\ x + y + z, \end{aligned}$$

and they all lie in the $[C_4^{\times 2} \rtimes S_3/C_6]$ orbit.

Remark 4.4. Edge wrote an entire paper about this quartic in 1938 [Edg38], in which he investigates its determinantal forms. Explicit equations for each of the 28 bitangents is included in [Edg38, p. 14], though he attributes this computation to Cayley. In [Edg38, §14], Edge extends his computations of bitangents and Cayley octads for the Dyck quartic to a more general family of quartics, mainly those of the form

$$(1 + \gamma^2)^2(x^4 + y^4 + z^4) - 2(1 + \gamma^4)(y^2z^2 + x^2y^2 + x^2z^2) = 0.$$

When $\gamma = \zeta_8$ is a primitive 8th root of unity, we obtain the Dyck quartic. For other values of γ , quartics of this shape are called *Edge quartics*. The Edge quartic with $\gamma = \pm\sqrt{5}$, from Example 1.2, was studied in [PSV11, p. 713]; we note via our classification the automorphism group is different — this Edge quartic is a Type IV quartic with $a = -\frac{34}{25}$.

4.3. Type III: Order 48. Order 48, structure $C_4 \odot A_4$.

There is a unique Type III quartic given by the equation:

$$x^4 + y^4 + z^4 + (4\zeta_3 + 2)x^2y^2,$$

with automorphism group given by

$$C_4 \odot A_4 = \left\langle \left(\begin{array}{ccc} \frac{1+i}{2} & \frac{-1+i}{2} & 0 \\ \frac{1+i}{2} & \frac{1-i}{2} & 0 \\ 0 & 0 & \zeta_3 \end{array} \right), \left(\begin{array}{ccc} \frac{1+i}{2} & \frac{-1-i}{2} & 0 \\ -\frac{1+i}{2} & -\frac{1-i}{2} & 0 \\ 0 & 0 & \zeta_3^2 \end{array} \right) \right\rangle \leq \mathrm{PGL}_3(\mathbb{C}).$$

A GAP computation shows that the 28 bitangents break into two orbits of sizes 24 and four. The first orbit has isotropy $C_2 \leq C_4 \odot A_4$, which is the non-normal cyclic subgroup of order two in A_4 , while the second orbit has isotropy given by the unique subgroup of order 12, which is cyclic:

Computation 4.5. The bitangents to a Type III quartic have orbits

$$[C_4 \odot A_4/C_2] + [C_4 \odot A_4/C_{12}].$$

There are no real points, so we unfortunately cannot visualize this example.

4.4. Type IV: The pencil of octahedral quartics. Order 24, structure S_4

Up to change of coordinates, Type IV quartics are given by the following pencil, which we call the pencil of *octahedral quartics*

$$x^4 + y^4 + z^4 + a(x^2y^2 + y^2z^2 + z^2x^2),$$

for $a \neq \frac{3}{2}(-1 \pm \sqrt{-7})$ (this would give the Klein) and $a \neq 0$ (this would give the Dyck quartic). As a varies, we sweep out a pencil of quartics, first studied by Wiman and Ciani independently. A modern discussion of this pencil of quartics can be found in [DK93, §8]. The paper [RGA97] is dedicated solely to the study of this pencil, where the authors study (among other things) the Fuschian uniformizations and the Jacobians of these quartics.

A Type IV quartic has automorphism group isomorphic to the symmetric group S_4 . For any such a as above, we see the automorphism group is

$$S_4 \cong \left\langle \left(\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right), \left(\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \right\rangle \leq \mathrm{PGL}_3(\mathbb{C}).$$

Computation 4.6. The bitangents of a Type IV quartic have orbits

$$[S_4/C_2^o] + [S_4/C_2^e] + [S_4/S_3].$$

Here C_2^o is an *odd* copy of $C_2 \leq S_4$, i.e. it is a single transposition, while C_2^e is a product of two disjoint transpositions, hence it is *even*.

Example 4.7. When $a = -3$, we compute there are 16 real bitangents, forming the orbits $[S_4/C_2^o]$ and $[S_4/S_3]$. We visualize this in Figure 4.

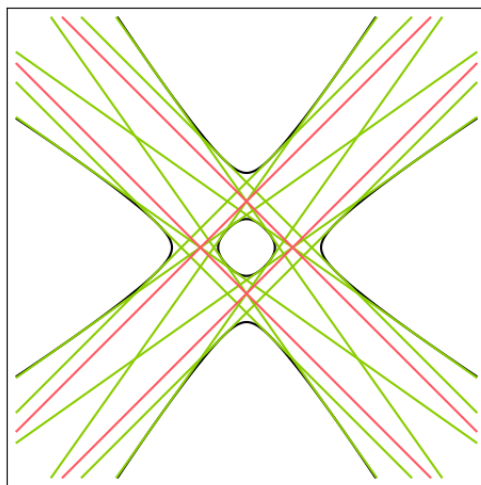


FIGURE 4. The orbits of the real bitangents on the Type IV quartic with $a = -3$. Lines in green are in the $[S_4/C_2^o]$ orbit, and lines in pink are in the $[S_4/S_3]$ orbit.

4.5. **Type V: Order 16.** Order 16, structure $P = C_4 \otimes K_4$ the *Pauli group*.

Notation 4.8. The automorphism group of a type V surface is the *Pauli group*, which we denote by P . In particular there are three non-central conjugacy classes of cyclic subgroups of order two, which we denote by $C_2^{(1)}$, $C_2^{(2)}$, and $C_2^{(3)}$, and the center is a cyclic group of order four which we denote by C_4^Z .

Type V quartics are given by the equation

$$x^4 + y^4 + z^4 + ax^2y^2$$

for $a \notin \{0, \pm 2\sqrt{-3}, \pm 6\}$. Their automorphism groups are of the form

$$P \cong \left\langle \left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \right\rangle \leq \text{PGL}_3(\mathbb{C}).$$

Computation 4.9. Any quartic of Type V has bitangents with orbits

$$[P/C_2^{(1)}] + [P/C_2^{(2)}] + [P/C_2^{(3)}] + [P/C_4^Z].$$

Example 4.10. When $a = -4$ we see eight real bitangents, as in Figure 5.

4.6. **Type VI: Order 9.** Order 9, structure C_9 .

There is a unique Type VI quartic, given by the equation

$$x^4 + xy^3 + yz^3,$$

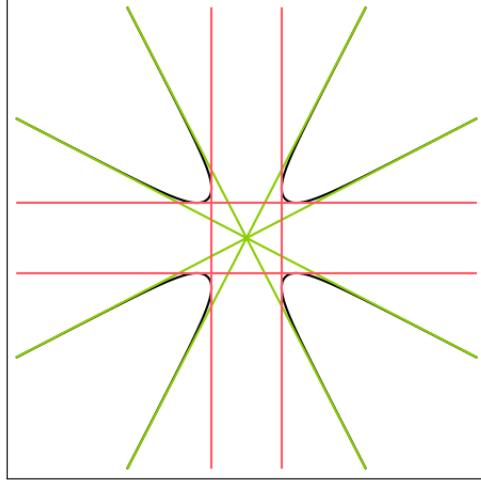


FIGURE 5. The orbits of the real bitangents on a Type V quartic. The four in green form the the orbit whose isotropy is the central C_4^Z . The four lines in pink are part of an orbit with $C_2^{(1)}$ isotropy.

whose automorphism group is cyclic of order nine:

$$C_9 \cong \left\langle \left(\begin{array}{ccc} \zeta_3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_9 \end{array} \right) \right\rangle \leq \mathrm{PGL}_3(\mathbb{C}).$$

Computation 4.11. The Type VI quartic has bitangents with orbits

$$3[C_9/e] + [C_9/C_9].$$

This quartic has four real lines, one in each orbit, in Figure 6.

4.7. Type VII: Order 8. Order 8, structure D_8 .

Type VII quartics are given by the equation

$$x^4 + y^4 + z^4 + ax^2y^2 + bxyz^2$$

for $b \neq 0$. The automorphism group is given by

$$D_8 \cong \left\langle \left(\begin{array}{ccc} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \right\rangle \leq \mathrm{PGL}_3(\mathbb{C}).$$

Writing the first generator as r and the second as s , we have three conjugacy classes of subgroups of order two, namely

$$\begin{aligned} C_2^{(1)} &:= \langle s \rangle \cong \langle r^2s \rangle \\ C_2^{(2)} &:= \langle rs \rangle \cong \langle r^3s \rangle \\ C_2^Z &:= \langle r^2 \rangle, \end{aligned}$$

where the superscript Z denotes that it is central.

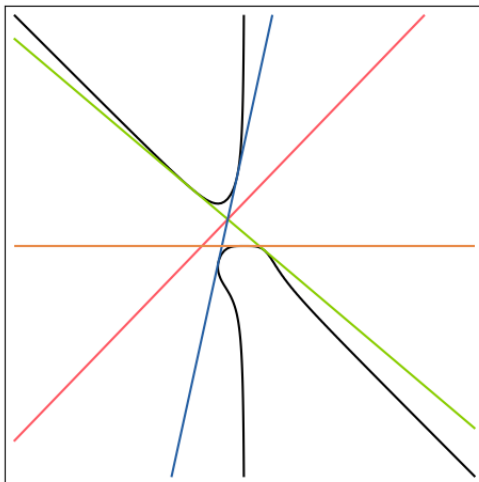


FIGURE 6. The orbits of real bitangents on the quartic of Type VI. In green are lines in the first trivial isotropy orbit. In pink are lines in the 2nd trivial isotropy orbit. In blue are lines in the 3rd trivial isotropy orbit. Finally in orange are lines in the $[C_9/C_9]$ orbit.

Computation 4.12. The bitangents of any Type VII quartic have orbits

$$[D_8/e] + 2[D_8/C_2^{(1)}] + 2[D_8/C_2^{(2)}] + [D_8/C_2^Z].$$

Example 4.13. For $a = -3$ and $b = 1$, we see eight real bitangents, pictured in Figure 7.

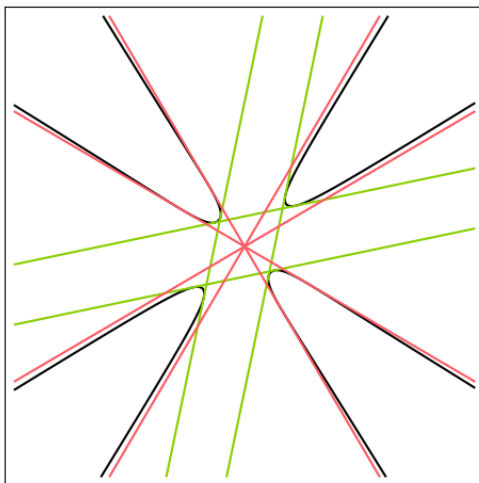


FIGURE 7. The orbits of real bitangents on an example quartic of Type VII. The green lines have trivial isotropy, i.e. they live in the $[D_8/e]$ orbit, while the pink -lines form the $[D_8/C_2^Z]$ orbit.

4.8. **Type VIII: Order 6.** Order 6, structure C_6 .

Type VIII quartics are given by

$$x^4 + y^4 + ax^2y^2 + yz^3$$

for $a \neq 0$. Its automorphism group is given by

$$C_6 \cong \left\langle \left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \end{array} \right) \right\rangle \leq \mathrm{PGL}_3(\mathbb{C}).$$

Computation 4.14. The bitangents of any Type VIII quartic have orbits

$$4[C_6/e] + [C_6/C_2] + [C_6/C_6].$$

Example 4.15. When $a = -3$, we see four real bitangents, two occurring in distinct orbits with trivial isotropy, one in the orbit with C_3 isotropy, and one fixed under the action of the automorphism group, pictured in Figure 8.

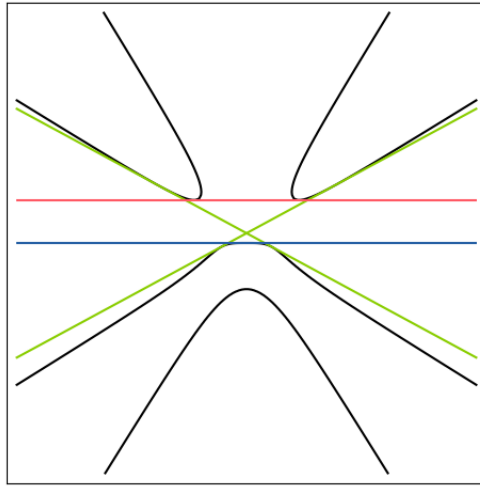


FIGURE 8. The orbits of real bitangents on an example quartic of Type VIII. In green are lines in the $[C_6/e]$ orbit. In pink are lines in the $[C_6/C_2]$ orbit. In blue are lines in the $[C_6/C_6]$ orbit.

4.9. Type IX: Order 6. Order 6, structure S_3

Type IX quartics are given by the equation

$$x^3z + y^3z + x^2y^2 + axyz^2 + bz^4$$

for $a \neq 0$. Its automorphism group is given by

$$S_3 \cong \left\langle \left(\begin{array}{ccc} \zeta_3 & 0 & 0 \\ 0 & \zeta_3^2 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array} \right) \right\rangle \leq \mathrm{PGL}_3(\mathbb{C}).$$

Computation 4.16. The bitangents on any Type IX quartic are given by

$$3[S_3/e] + 3[S_3/C_2] + [S_3/S_3].$$

Example 4.17. When $a = -25$ and $b = 10$, we compute eight real bitangents, pictured in Figure 9.

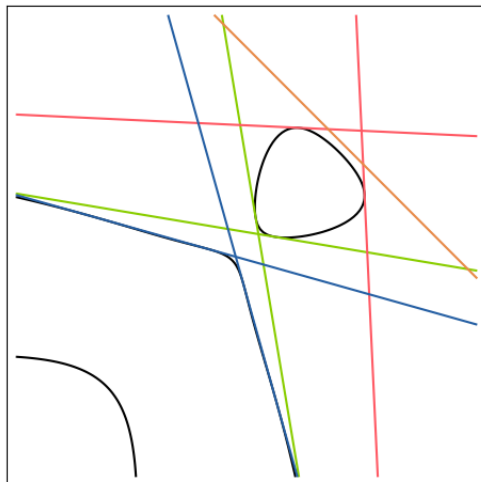


FIGURE 9. The orbits of real bitangents on an example quartic of Type IX. In green, pink, and blue are lines in different $[S_3/e]$ orbits. The line in orange lies in an $[S_3/C_2]$ orbit. Not pictured is a bitangent $z = 0$ at infinity which forms the $[S_3/S_3]$ orbit.

4.10. Type X: Order 4. Order 4, structure K_4

Type X quartics are given by the equation

$$x^4 + y^4 + z^4 + ax^2y^2 + by^2z^2 + cx^2z^2$$

for a, b, c all distinct. Its automorphism group is given by

$$K_4 \cong \left\langle \left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right), \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right\rangle \leq \mathrm{PGL}_3(\mathbb{C}).$$

Computation 4.18. The bitangents on any Type X quartic have orbits

$$4[K_4/e] + 2[K_4/C_2^L] + [2K_4/C_2^A] + 2[K_4/C_2^R].$$

Example 4.19. When $(a, b, c) = (-9, -3, -8)$, we compute 16 real bitangents, forming four orbits with trivial isotropy, pictured in Figure 10.

4.11. Type XI: Order 3. Order 3, structure C_3

Type XI quartics are given by the equation

$$x(x-y)(x-ay)(x-by) + yz^3,$$

for a, b such that $a \neq 1$, $b \neq 1 - a$, and $(x-a)(x-b) \neq x^2 + x + 1$. It has automorphism group

$$C_3 \cong \left\langle \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \zeta_3 \end{array} \right) \right\rangle \leq \mathrm{PGL}_3(\mathbb{C}).$$

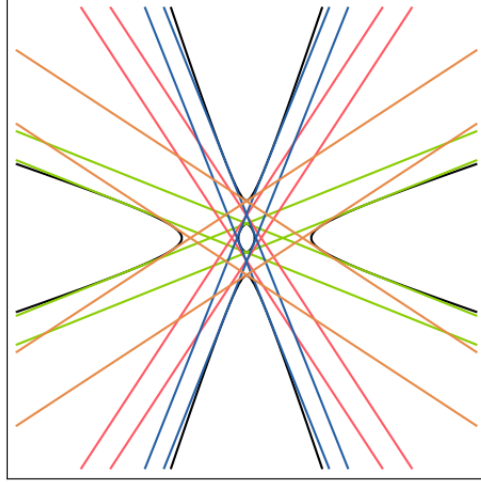


FIGURE 10. The orbits of real bitangents on an example quartic of Type X. There are 16 bitangents, forming four orbits of the form $[K_4/e]$.

Computation 4.20. The bitangents on any Type XI quartic have orbits

$$9[C_3/e] + [C_3/C_3].$$

Example 4.21. When $a = 2$ and $b = 3$, we compute four real bitangents, pictured in Figure 11.

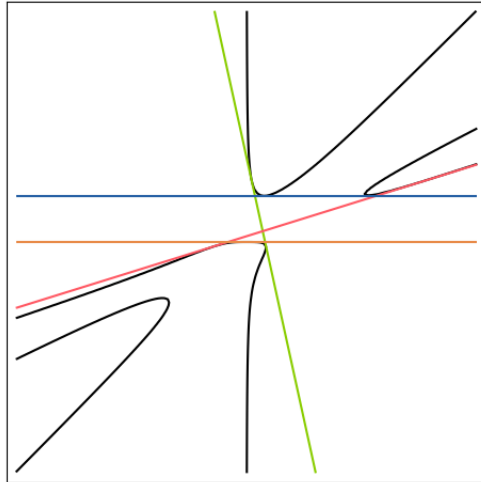


FIGURE 11. The orbits of real bitangents on an example quartic of Type XI. Orbits with colors green, pink, and blue have trivial isotropy. The orange line is in a $[C_3/C_3]$ orbit, i.e. it is fixed by the automorphism group.

4.12. **Type XII: Order 2.** Order 2, structure C_2 .

All Type XII quartics are given by

$$x^4 + y^4 + z^4 + x^2(ay^2 + byz + cz^2) + dy^2z^2,$$

for $b \neq 0$, $a \neq -2$, and $c \neq -d$. Note that if $b = 0$ we obtain a Type X quartic, and if $a = -2$, $b = 0$, and $c = -d$ we obtain a Type IX curve.

This has automorphism group

$$C_2 \cong \left\langle \left(\begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \right\rangle \leq \mathrm{PGL}_3(\mathbb{C}).$$

Computation 4.22. The bitangents on any Type XII quartic have orbits

$$12[C_2/e] + 4[C_2/C_2].$$

Example 4.23. For the Type XII quartic defined by the equation

$$x^4 + x^2(-y^2 - 4z^2) + y^4 - y^2z^2 - z^4,$$

we have four real bitangents, forming three orbits, pictured in Figure 12.

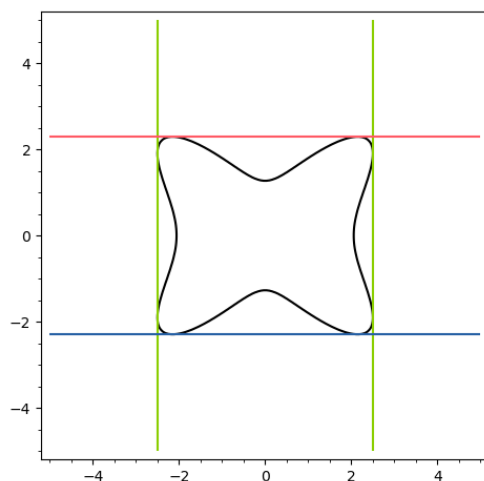
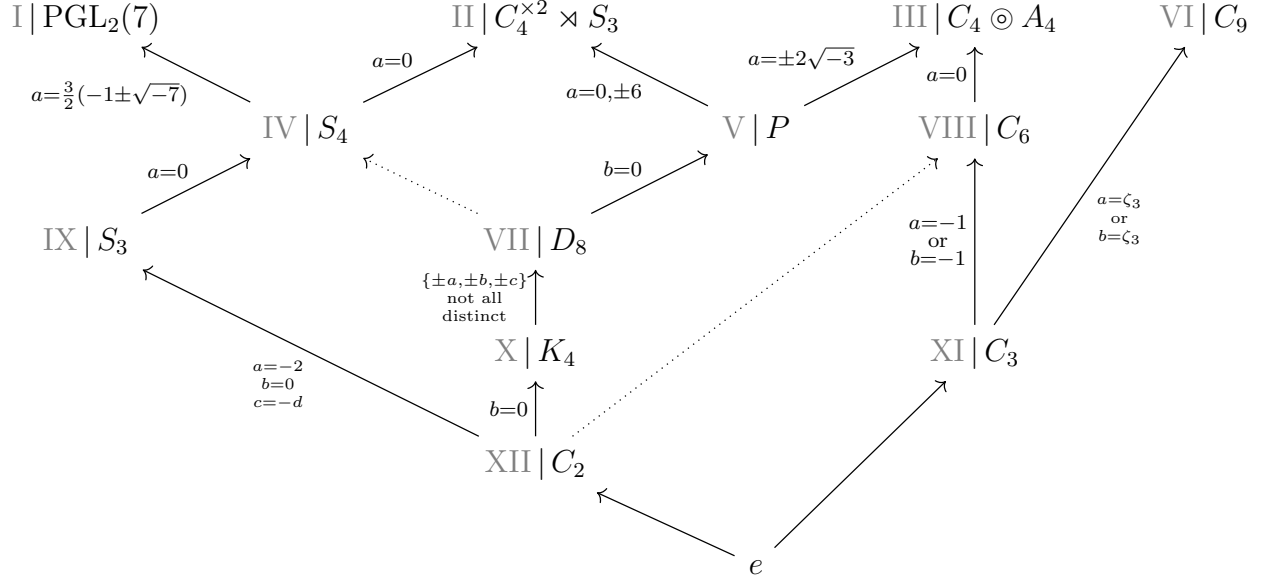


FIGURE 12. The orbits of real bitangents on an example quartic of Type XII. Lines in green have trivial isotropy, while the lines in pink and blue are fixed by the automorphism group.

5. THE SUBGROUP LATTICE AND RESTRICTION

As mentioned in Remark 3.8, we may restrict our attention to a subgroup $H \leq G \cong \mathrm{Aut}(C)$ of automorphisms for a given curve, and the action of that group on the bitangents is the restriction of the G -set of bitangents to the subgroup H . As we have seen in Theorem 2.4, we have families of curves attached to certain values of a , b , etc., and when these take particular values the type of the quartic can change. Often how this occurs is that the entire family is invariant under the automorphism group, however for a special value of a coefficient, a new polynomial relation holds, introducing an extra automorphism to the equation and enlarging the automorphism group of the curve. To that end, it is important to understand which automorphism groups are subgroups of one another.

We therefore copy a table from [Hen76] demonstrating subgroup inclusion. For each entry we include both the type and automorphism group, and, where relevant, we label the inclusions with the special values from the example forms of quartics provided in Theorem 2.4.



The groups as we've presented them may not be literally subgroups of one another, a projective change of basis may be needed. We explain the meaning behind the dashed arrows in Remark 5.4.

Example 5.1. Some lovely examples of the phenomenon we're discussing can be found in [Dol12, pp. 269–270]; we include one such example here, where we show how to derive the special values $a = -6, 0, 6$ where the pencil of Type V quartics becomes a Dyck quartic. Recall the equation for a Type V quartic:

$$x^4 + y^4 + z^4 + ax^2y^2.$$

It is clear that by setting $a = 0$ we obtain the Dyck quartic, but where do the other values $a = \pm 6$ come from? We see that the Pauli group P is not literally a subgroup of $C_4^{\times 2} \rtimes S_3$ as presented, nevertheless $C_4^{\times 2} \rtimes S_3$ contains a unique conjugacy class of subgroups isomorphic to P . This is reflected via a change of basis matrix (unique up to conjugacy in $C_4^{\times 2} \rtimes S_3$ — note there are three conjugacy classes of P in $C_4^{\times 2} \rtimes S_3$)

$$g = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 8^{1/4} \end{pmatrix}.$$

Conjugating P by this change of basis, we see that $gPg^{-1} \leq C_4^{\times 2} \rtimes S_3$, and the equation for a Type V quartic changes to be

$$(x + y)^4 + (x - y)^4 + 8^{1/4}z^4 + a(x + y)^2(x - y)^2 = (a + 2)x^4 + (a + 2)y^4 + 8z^4 + (12 - 2a)x^2y^2.$$

We can now ask for which values of a does such a quartic satisfy the $C_4^{\times r} \rtimes S_3$ symmetries in Equation 2. The first generator cycles the variables, for which we see we need $12 - 2a = 0$ for

the x^2y^2 term to vanish, i.e. we need $a = 6$. For this choice of a we satisfy the full symmetry group $C_4^{\times 2} \rtimes S_3$ and hence obtain the Type II quartic. The case $a = -6$ is similar, but using a conjugate change of basis.

Remark 5.2. We remark this line of reasoning is basically equivalent to being aware of clever relations for working with homogeneous polynomials and their vanishing loci. Dolgachev remarks that the constraint above comes from knowing the identity

$$x^4 + y^4 = \frac{1}{8} \left((x+y)^4 + (x-y)^4 + 6(x+y)^2(x-y)^2 \right).$$

Leveraging identities such as these in projective geometry is characteristic of much of algebra in the latter half of the 19th century.

What happens when this line of reasoning fails? Consider the following example:

Proposition 5.3. There are no special values of a, b, c, d for which a Type XII quartic becomes a Type VIII quartic.

Proof. Since C_6 is abelian and C_2 is a subgroup as written, we just have to check whether any special values of a, b, c, d exist for which a Type XII quartic can satisfy a sixfold symmetry. Applying the generator for C_6 to a Type XII quartic, we obtain

$$x^4 + y^4 + \zeta_3 z^4 + x^2(ay^2 + b\zeta_3 yz + c\zeta_3^2 z^2) + d\zeta_3^2 y^2 z^2,$$

and there are no values of a, b, c, d for which this is equal to the original equation.⁷ \square

Remark 5.4. This is precisely the meaning of the dashed arrows in our subgroup diagram. In the language of the moduli of symmetric quartics, this means the Type VIII quartics don't lie on the boundary of the Type XII quartics, but rather they share a common boundary at the Type III quartics. The argument for the dashed arrow from Type VII to Type IV is omitted.

5.1. Leveraging Euler classes to explore restricted actions. We've seen that the locus of D_8 -symmetric quartics is not glued to the locus of S_4 -symmetric quartics, so from the naive perspective of the moduli space, we shouldn't expect that there is much coherence between how these groups act on bitangents. Nevertheless we can see the following example:

Example 5.5. Consider the Edge quartic (of Type IV) with its restricted $D_8 \leq S_4$ -symmetry on the xy -plane. The D_8 -orbits of the lines are graphed in Figure 13

This is the same count as the D_8 -action on the bitangents of a Type VII quartic (Computation 4.12), despite the fact that the Edge quartic cannot be obtained by deforming a family of quartics with automorphism group D_8 . So the fact that we obtained the same answer above is perhaps not obvious from the moduli perspective. It is, however, obvious from the perspective of equivariant Euler classes!

Proposition 5.6. Let $H \leq G$ be a subgroup, and let $g \in G$ be arbitrary. Then we can consider restricting a G -set along the inclusion of H in G , or along the group homomorphism $c_g: H \rightarrow G$ given by $h \mapsto ghg^{-1}$. These induce naturally isomorphic functors $\text{Fin}_G \rightarrow \text{Fin}_H$.

⁷Note the existence of Type III quartics does not contradict this line of reasoning — the special values for passing from XII to X and then from X to VII are contradictory, so an intermediate change of basis is first needed, and new parameters need to be defined.

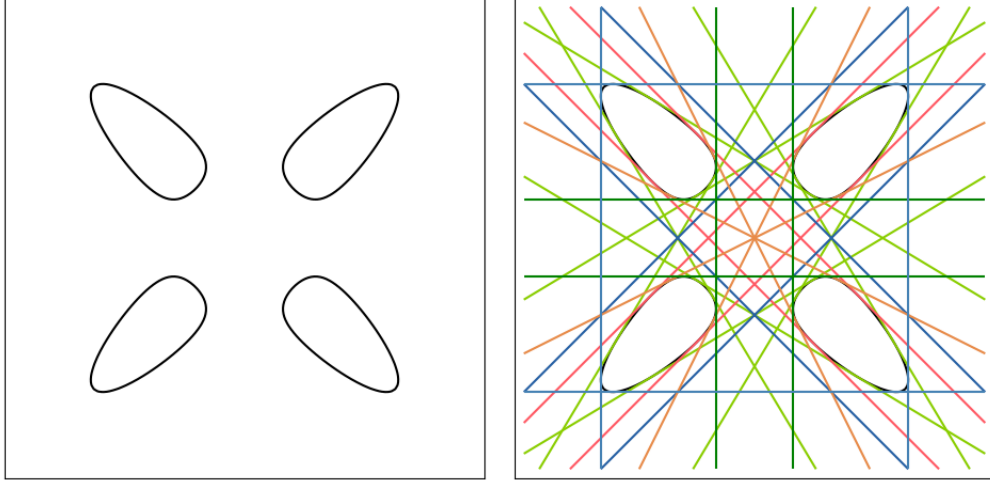


FIGURE 13. The *Edge quartic*, with D_8 -orbits on its bitangents given by $[D_8/e] + 2[D_8/C_2^{(1)}] + 2[D_8/C_2^{(2)}] + [D_8/C_2^Z]$.

Proof. It suffices to observe that conjugacy by g is a natural isomorphism on the category of finite G -sets. \square

Corollary 5.7. Let $E \rightarrow M$ be any G -equivariant enumerative problem satisfying the hypotheses in [Bra24, 1.1] (e.g. counting lines on symmetric cubic surfaces or counting bitangents to planar quartics). Then the restricted Euler class $n_H(E)$, obtained as the image of

$$\begin{aligned} \pi_0^G \text{MU}_G &\rightarrow \pi_0^H \text{MU}_H \\ n_G(E) &\mapsto n_H(E) \end{aligned}$$

depends only on the conjugacy class of H in G .

Proof. Equivariant Euler classes valued in homotopical bordism are in the injective image of the Burnside ring by [Bra24, 5.24], hence it suffices to prove the statement there. This follows directly from group completing isomorphism classes of finite G -sets and applying Proposition 5.6. \square

This explains what we observed in Example 5.5. Although there we cannot deform Type VII quartics into Type IV quartics, there are unique conjugacy classes of D_8 and S_4 in PGL_3 occurring as the automorphism groups of quartics, and the copy of D_8 is subconjugate to S_4 .

Remark 5.8.

- (1) This means that we can still restrict computations along the dashed arrows in our subgroup lattice, despite the fact that the associated components in the moduli are not attached directly to one another.
- (2) This line of reasoning fails for isomorphic subgroups which are not conjugate — given an S_4 -symmetric cubic surface, the restricted action on its lines via the two non-conjugate cyclic subgroups of order two are different, as computed in [Bra24, 6.5].

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