MONODROMY IN THE SPACE OF SYMMETRIC CUBIC SURFACES WITH A LINE

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Abstract. We study the symmetric enumerative problem of finding a line on a symmetric cubic surface. We show that this problem has Galois group isomorphic to the Klein 4-group. Additionally, we prove that the moduli space of symmetric cubic surfaces is an arithmetic quotient of the complex hyperbolic line. Our proofs use techniques from equivariant line geometry, Hodge theory, and homotopy continuation.

1. Introduction

The *Galois group* of an enumerative problem is a classical object of study in enumerative algebraic geometry. It was first introduced by Jordan as one of the main subjects of interest at the genesis of Galois theory [Jor70]. This idea enjoyed a revival a century later when Harris proved that the Galois group of an enumerative problem agrees with the monodromy group of its associated cover [Har79]. In modern mathematics Galois groups can be approached from a wide number of perspectives, from Hodge theory and hyperbolic geometry [ACT02], to Lie theory [Man06], to numerical analysis and homotopy continuation [LS09], to name a few.[1]

Contemporary geometers such as Klein were interested in exploring how symmetries of objects manifest in enumerating various quantities attached to them. Recent work of the first-named author introduces tools from equivariant homotopy theory to explore how Poncelet's principle of conservation of number interacts with symmetry, an example being that a smooth cubic surface defined by a symmetric polynomial always has the same S_4-symmetries on its lines [Bra24]. Such cubic surfaces are called *symmetric cubic surfaces*.

In this paper we initiate an exploration of Galois groups of symmetric enumerative problems. This flavor of question is well-studied in geometric group theory; for example, many have studied rigidity phenomena for finite index subgroups of lattices in Lie groups (e.g. [Mar91] and [FW08]) and equivariant problems for their non-linear analogues like mapping class groups and $\mathrm{Out}(F_n)$ (e.g. [BH73], [MH75], [FH07], and [LLS24]). However the setting we pursue is of a completely different shape — since the Galois group of lines on a cubic surface (and many related problems) is finite, we cannot leverage such tools, e.g. Teichmüller theory, to approach this question, and alternative techniques are needed.

Our main result is a *computation of the Galois group of lines on symmetric cubic surfaces*, which we show is equal to the Klein 4-group. This is carried out via a combination of moduli-theoretic techniques, classical analysis of the Weyl group of the E_6 lattice, as well as group-theoretic

[1] For a lovely introduction to the history and appearance of Galois groups in enumerative geometry, we refer the reader to [SY21].

computations in GAP and contemporary certified tracking homotopy continuation algorithms. Along the way we prove that *the moduli of stable symmetric cubic surfaces is an arithmetic quotient of the complex hyperbolic line*. The latter result mirrors the landmark work of Allcock, Carlson, and Toledo at the turn of the century [ACT02], where they show the moduli space of stable cubic surfaces is an arithmetic quotient of complex hyperbolic 4-space. We explore the appearance of our Klein 4-group K_4 in both the Weyl group of E_6 and in the projective orthogonal group $\mathrm{PO}(4,1,\mathbb{F}_3)$.

1.1. **Main results.** Before we state our main theorems more formally, we fix some notation. Let $\mathcal{M}$ (resp. $\mathcal{M}^s$) denote the moduli space of smooth (resp. stable) cubic surfaces. Similarly, let $\mathcal{S}$ (resp. $\mathcal{S}^s$) denote the moduli space of smooth (resp. stable) symmetric cubic surfaces. Finally, let $\mathcal{H}^{S_4} \subset \mathbb{CH}^1$ denote the symmetric discriminant locus of the period map.

Theorem 1.1. There are analytic isomorphisms of orbifolds $\mathcal{S} \cong P\Gamma\backslash(\mathbb{CH}^1 - \mathcal{H}^{S_4})$ and $\mathcal{S}^s \cong P\Gamma\backslash\mathbb{CH}^1$, where $\Gamma < \mathrm{U}(1,1)$ is an arithmetic lattice. Moreover the inclusion of moduli spaces $\mathcal{S} \to \mathcal{M}$ is compatible with the embedding of locally symmetric orbifolds $P\Gamma\backslash\mathbb{CH}^1 \to P\hat{\Gamma}\backslash\mathbb{CH}^4 \cong \mathcal{M}^s$.

For the precise statement of Theorem 1.1, its semistable extension, and its proof, see Theorem 4.11. Roughly, the idea behind the proof is to record the S_4-action in the period data and use the S_4-invariant subspace to define the period domain associated to symmetric cubic surfaces. We also determine the arithmetic group Γ explicitly in Proposition 4.9.

Let $\widetilde{\mathcal{M}}$ (resp. $\widetilde{\mathcal{S}}$) denote the space of (resp. symmetric) cubic surfaces equipped with a line. Recall that Jordan showed that the connected 27 lines cover $\widetilde{\mathcal{M}} \to \mathcal{M}$ has Galois group $W(E_6)$, the Weyl group of E_6. Allcock–Carlson–Toledo recovered this fact Hodge-theoretically by considering an appropriate congruence cover of their uniformized moduli space $P\hat{\Gamma}\backslash\mathbb{CH}^4$ and using the exceptional isomorphism $W(E_6) \cong \mathrm{PO}(4,1,\mathbb{F}_3)$. The following monodromy group result is an equivariant analog of Jordan's theorem for the symmetric 27 lines cover $\widetilde{\mathcal{S}} \to \mathcal{S}$:

Theorem 1.2. The (disconnected) symmetric 27 lines cover $\widetilde{\mathcal{S}} \to \mathcal{S}$ has monodromy group isomorphic to the Klein 4-group $K_4 < W(E_6)$.

The action of K_4 on 27 labeled lines is explicitly worked out as permutations in Data A.4. This allows us to completely characterize the covering space $\widetilde{\mathcal{S}}$ — it has 12 connected components, with each one corresponding to an explicit K_4-set; see Corollary 5.16 for details.

There are a few reasons why Theorem 1.2 is surprising. First off, the symmetric group S_4 and symmetric monodromy group K_4, thought of as subgroups of $W(E_6)$, intersect trivially — this means that if we want to witness the S_4-action on a symmetric cubic surface through monodromy, we must leave the symmetric locus in the total moduli space. Second, a number of restrictions coming from Hodge theory constrain the monodromy group to a group of order 96. However, these restrictions provably do not suffice, as we can name explicit elements of this restricted subgroup that cannot arise via symmetric monodromy. This stands in direct contrast with reasoning used when studying similar problems, such as in [ACT10, Section 8].

1.2. **Paper structure.** In Section 2, we review the construction of the (marked) moduli space of cubic surfaces, and explicitly realize the moduli space of symmetric cubic surfaces as a GIT quotient. In Section 3 we give an overview of the work of Allcock–Carlson–Toledo, including fundamental facts about framed cubic surfaces and their associated period maps, which yields their main theorem, a uniformization of the moduli of cubic surfaces by complex hyperbolic 4-space. In Section 4, we analogously define a period map for symmetric cubic surfaces via symmetric framings of cubic surfaces, and uniformize the moduli space of symmetric cubic surfaces by the complex hyperbolic line. This section finishes with a Hodge theoretic restriction of symmetric monodromy to a group of order 96, which it turns out will properly contain the symmetric monodromy group. In Section 5 we determine that the symmetric monodromy group is the Klein 4-group K_4, and we describe how it acts on the 27 lines. Moreover, we show that the 27 lines cover over the symmetric locus $\widetilde{S} \to \mathcal{S}$ splits into 12 connected components, and explicitly determine the K_4-set structure associated to each connected component. Finally, Section 6 describes how to alternatively witness the symmetric monodromy group K_4 in $W(E_6)$ and $\mathrm{PO}(4,1,\mathbb{F}_3)$ via representation theoretic constructions. Appendix A contains explicit data regarding how the symmetric group S_4 and the generators of the symmetric monodromy group K_4 act on the 27 lines on the Fermat cubic surface.

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1.4. **Notation.** We use `\mathcal` letters to indicate *parameter spaces*, being both vector spaces parametrizing polynomials and moduli spaces of their vanishing loci.

notation	**meaning**
$\mathcal{W}$	$\mathbb{C}[x_0, \ldots, x_3]_{(3)}$
$\mathcal{V}$	$\mathbb{C}[x_0, \ldots, x_3]_{(3)}^{S_4}$
$\mathcal{S}$	moduli of symmetric cubic surfaces
$\mathcal{M}$	moduli of cubic surfaces
decoration	**meaning**
$(-)^{\mathrm{sm}}$ or no decoration	moduli of smooth objects
$(-)^{s}$	moduli of stable objects
$(-)^{ss}$	moduli of semistable objects
$\widetilde{(-)}$	moduli of cubic surfaces equipped with a line
$\widehat{(-)}$	moduli of marked cubic surfaces

2. Moduli constructions

The content of this section is standard and well-known — see [Zhe21] for example. We will review how to construct the moduli space of (marked) smooth cubic surfaces as a GIT quotient, and then analogously construct the moduli space of (marked) symmetric cubic surfaces. We end this section with a quick discussion of stability and semistability of cubic surfaces, concluding with the facts that the Cayley nodal cubic is the only non-smooth symmetric stable cubic surface, and that the tricuspidal cubic is the only non-stable symmetric semistable cubic surface.

2.1. **Parameter space of cubic surfaces.** Let $\mathcal{W} = \mathbb{C}[z_0, z_1, z_2, z_3]_{(3)}$ denote the 20-dimensional vector space of degree 3 homogeneous polynomials in 4 variables. Every $f \in \mathcal{W}\backslash\{0\}$ defines a cubic surface $Z(f)$ in $\mathbb{P}^3$, and two elements f_1 and f_2 determine the same surface $Z(f_1) = Z(f_2)$ if and only if $f_1 = \lambda f_2$ for some $\lambda \in \mathbb{C}^*$. Thus $\mathbb{P}(\mathcal{W}) \cong \mathbb{P}^{19}$ can be naturally thought of as the parameter space of cubic surfaces in $\mathbb{P}^3$.

There is a linear action of $\mathrm{SL}(4,\mathbb{C})$ on $\mathcal{W}^{\mathrm{sm}}$ given by $g \cdot f := f \circ g^{-1}$. This induces a left action of $\mathrm{PGL}(4,\mathbb{C})$ on the projectivization $\mathbb{P}(\mathcal{W})$.

Definition 2.1. Consider the left $\mathrm{SL}(4,\mathbb{C})$-action on $\mathcal{W}$ induced by permuting coordinates on $\mathbb{P}^3$. For $f \in \mathcal{W}$, we say that f is

(1) *smooth* if its associated cubic surface is smooth,
(2) *stable* if the orbit $\mathrm{SL}(4,\mathbb{C}) \cdot f$ is closed, and the stabilizer subgroup is finite,
(3) *semistable* if 0 is not in the closure of the orbit $\mathrm{SL}(4,\mathbb{C}) \cdot f$.

We denote by $\mathcal{W}^{\mathrm{sm}}$ (respectively $\mathcal{W}^s$, and $\mathcal{W}^{ss}$) the subsets of $\mathcal{W}$ corresponding to smooth cubic surfaces (respectively stable, and semistable). It is classically known that we have containments

$$\mathcal{W}^{\mathrm{sm}} \subseteq \mathcal{W}^s \subseteq \mathcal{W}^{ss}.$$

The action of $\mathrm{SL}(4,\mathbb{C})$ on each of these loci extends to an action of $\mathrm{PGL}(4,\mathbb{C})$ on their projectivizations. We can take the respective GIT quotients to construct various moduli spaces of cubic surfaces.

Definition 2.2. We denote by

$$\mathcal{M}^{\mathrm{sm}} := \mathrm{PGL}(4,\mathbb{C})\backslash\!\backslash\mathbb{P}(\mathcal{W}^{\mathrm{sm}}),$$
$$\mathcal{M}^{s} := \mathrm{PGL}(4,\mathbb{C})\backslash\!\backslash\mathbb{P}(\mathcal{W}^{s}),$$
$$\mathcal{M}^{ss} := \mathrm{PGL}(4,\mathbb{C})\backslash\!\backslash\mathbb{P}(\mathcal{W}^{ss}),$$

the *moduli space of smooth/stable/semistable cubic surfaces.*

Convention 2.3. When we write a moduli space without a subscript, e.g. $\mathcal{M}$, we implicitly mean the moduli of smooth objects.

The following classical result characterizes stable and semistable cubic surfaces by their singularities.

Theorem 2.4 (Hilbert, [Hil93])**.** A cubic surface is stable if and only if it its singularities are ordinary nodes. A cubic surface is semi-stable if and only if its singularities are ordinary nodes or A_2 singularities.

Lemma 2.5 ([ACT02, 4.6])**.** The cubic form $z_0^3 - z_1z_2z_3$ is the unique closed $\mathrm{SL}(4,\mathbb{C})$-orbit of semistable non-stable cubic surfaces.

Since points in the GIT quotient $\mathcal{M}^{ss}$ correspond to closed orbits, this indicates that there is a unique point in the moduli space of semistable cubic surfaces corresponding to a point which is not stable. This is given by the unique *tricuspidal cubic surface*, defined by the equation mentioned, and pictured in Figure 1.

Let $\widetilde{\mathcal{W}^{\mathrm{sm}}}$ denote the parameter space of smooth cubic forms equipped with an incident line. Concretely, this is the incidence variety

$$\widetilde{\mathcal{W}^{\mathrm{sm}}} = \{(f, \ell) \in \mathcal{W}^{\mathrm{sm}} \times \mathrm{Gr}(2,4) : \ell \subset Z(f)\}.$$

The GIT quotient $\widetilde{\mathcal{M}} = \mathrm{PGL}(4,\mathbb{C}) \backslash\!\backslash \mathbb{P}\widetilde{\mathcal{W}^{\mathrm{sm}}}$ is the moduli space of smooth cubic surfaces equipped with a line. Since the natural projection $\widetilde{\mathcal{W}^{\mathrm{sm}}} \to \mathcal{W}^{\mathrm{sm}}$ is $\mathrm{PGL}(4,\mathbb{C})$-equivariant, it descends to a map of moduli spaces $\widetilde{\mathcal{M}} \to \mathcal{M}$.

2.2. **Marked parameter space of cubic surfaces.** Recall that a free finitely generated $\mathbb{Z}$-module L equipped with an integral symmetric (resp. symplectic) non-degenerate bilinear form q defines a *symmetric (resp. symplectic) lattice structure* (L, q). The lattice structure on the intersection form of a smooth cubic surface is classically obtained by viewing the surface as a blowup of the projective plane at six points.

Proposition 2.6. Let $X = V(f) \subset \mathbb{P}^3$ denote a smooth cubic surface determined by some cubic form $f \in \mathcal{W}^{\mathrm{sm}}$. Then $H = H^2(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 7, and the cup product $\langle \cdot, \cdot \rangle$ determines a signature $(1,6)$ symmetric unimodular lattice structure $(H, \langle \cdot, \cdot \rangle)$.

Let $\eta_X \in H$ denote the canonical class on X and $(L, q) \cong \langle 1 \rangle \oplus \langle -1 \rangle^{\oplus 6}$ be an abstract lattice isomorphic to $(H, \langle \cdot, \cdot \rangle)$. Fix some $\eta \in L$ so that $(L, q, \eta) \cong (H, \langle \cdot, \cdot \rangle, \eta_X)$.

Definition 2.7. A *marking* of a smooth cubic surface X is an isomorphism of lattices

$$\phi : (H, \langle \cdot, \cdot \rangle, \eta_X) \to (L, q, \eta).$$

We say a cubic form with marking (f_1, ϕ_1) is equivalent to the pair (f_2, ϕ_2) if there exists some $g \in \mathrm{PGL}(4,\mathbb{C})$ so that $g(f_1) = f_2$ and $\phi_2 = \phi_1 \circ g^*$. We will let $\widehat{\mathcal{W}^{\mathrm{sm}}}$ denote the parameter space of marked smooth cubic forms, which is naturally a complex manifold [ACT02, 3.2].

Let $\widehat{\mathcal{M}}$ denote the GIT quotient $\mathrm{PGL}(4,\mathbb{C}) \backslash\!\backslash \mathbb{P}\widehat{\mathcal{W}^{\mathrm{sm}}}$. We refer to this as the *moduli space of smooth marked cubic surfaces*. As cubic surfaces vary, their markings will vary as well. Since any two markings differ by an automorphism of the abstract lattice (L, q, η), we obtain a representation of the fundamental group of the moduli space of smooth marked cubic surfaces. The following is a relevant consequence of work of Beauville on monodromy in the universal family of degree d hypersurfaces which was classically known for cubic surfaces [Bea06]:

Proposition 2.8. The space $\widehat{\mathcal{M}}$ is a connected, Hausdorff complex manifold which is a covering space of $\mathcal{M}$. Moreover, the monodromy representation

$$\pi_1(\mathcal{M}, X) \to \mathrm{Aut}(H^2(X,\mathbb{Z}), \eta_X)$$

is surjective and has image isomorphic to the Weyl group of the root lattice E_6, denoted $W(E_6)$.

It is also classically known that the moduli space of marked cubic surfaces $\widehat{\mathcal{M}}$ is isomorphic to the moduli space of cubic surfaces equipped with a line $\widetilde{\mathcal{M}}$ [Bea09, pg. 19]. We shall freely identify these spaces.

2.3. **Parameter space of symmetric cubic surfaces.** Recall that a degree d homogeneous polynomial $f(z_1, \ldots, z_n)$ is *symmetric* if it is invariant under natural S_n-action on $z_1, \ldots, z_n$, i.e. $f(z_1, \ldots, z_n) = f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$ for all $\sigma \in S_n$. When $n > d$, the vector space $\mathbb{C}[z_1, \ldots, z_n]^{S_n}_{(d)}$ of symmetric homogeneous degree d polynomials in n variables is $p(d)$-dimensional, where $p(d)$ denotes the number of partitions of d. A basis will be denoted by $\{m_\alpha\}$, where m_α is a homogeneous symmetric polynomial indexed by the partitions $\alpha \vdash d$.

In the case of symmetric cubic forms in 4 variables, the vector space $\mathcal{V} := \mathcal{W}^{S_4}$ admits a basis of the form

$$m_3(z_0, z_1, z_2, z_3) = \sum z_i^3,$$
$$m_{21}(z_0, z_1, z_2, z_3) = \sum z_i^2 z_j,$$
$$m_{111}(z_0, z_1, z_2, z_3) = \sum z_i z_j z_k,$$

so any symmetric cubic form f in 4 variables can be uniquely written as a linear combination

$$f = a \cdot m_3 + b \cdot m_{21} + c \cdot m_{111}.$$

We see that the the parameter space of symmetric cubic forms $\mathbb{P}(\mathcal{V}) = \mathbb{P}^2$ embeds linearly into the parameter space of cubic forms $\mathbb{P}(\mathcal{W})$. Define Δ^{S_4} to be the symmetric discriminant curve, which corresponds to the locus of S_4-invariant singular cubic forms in the parameter space $\mathbb{P}\mathcal{V}$.

In order to form a GIT quotient parametrizing a moduli space of symmetric cubic surfaces, we need to understand how the action of $\mathrm{PGL}(4, \mathbb{C})$ preserves or fails to preserve the symmetry of the associated cubic surface. The following is a basic algebra fact that will be relevant to much of what follows:

Proposition 2.9. Let S_4 be a subgroup of any group G. Then the normalizer $N_G(S_4)$ is generated by S_4 and its centralizer $C_G(S_4)$.[2]

Proof. Given any $g \in N_G(S_4)$, we have $g\sigma g^{-1} \in S_4$ for all $\sigma \in S_4$. Thus conjugation by g defines an automorphism of S_4. Recall that S_n is a complete group for $n \neq 2, 6$, and so every automorphism of S_4 is an inner automorphism. This implies that for each $g \in N_G(S_4)$, there exists some $\eta \in S_4$ such that

$$g\sigma g^{-1} = \eta\sigma\eta^{-1} \Leftrightarrow \sigma = \eta^{-1} g\sigma g^{-1}\eta = \eta^{-1} g\sigma(\eta^{-1} g)^{-1}$$

for all $\sigma \in S_4$. Thus $\eta^{-1} g \in C_G(S_4)$, and so $g \in C_G(S_4) \cdot S_4$. This proves the claim. ☐

[2]The general fact we are using is that every automorphism of a complete group is inner. Thus a complete subgroup of any group has normalizer generated by the subgroup and its centralizer. The argument we give works *mutatis mutandis*.

Proposition 2.10. The normalizer of the permutation subgroup S_4 in $\mathrm{PGL}(4,\mathbb{C})$ is

$$N_{\mathrm{PGL}(4,\mathbb{C})}(S_4) \cong \left\{ \begin{pmatrix} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \\ 1 & 1 & 1 & \lambda \end{pmatrix} \cdot P : \lambda \notin \{1, -3\},\ P \in S_4 \leq \mathrm{PGL}(4,\mathbb{C}) \right\}$$

Proof. By Proposition 2.9, it suffices to determine the centralizer of S_4 in $\mathrm{PGL}(4,\mathbb{C})$. One can then calculate that the subgroup of permutation matrices in $\mathrm{GL}(4,\mathbb{C})$ is centralized by matrices of the form

$$C(a,b) = \begin{pmatrix} a & b & b & b \\ b & a & b & b \\ b & b & a & b \\ b & b & b & a \end{pmatrix}$$

where $b \neq a, a \neq -3b$.

Let $\varphi : \mathrm{GL}(4,\mathbb{C}) \to \mathrm{PGL}(4,\mathbb{C})$ be the projectivization homomorphism. Since the permutation matrices intersect the center of $\mathrm{GL}(4,\mathbb{C})$ trivially, we have $\varphi(S_4) \cong S_4$. To determine the rest of the image of $N_{\mathrm{PGL}(4,\mathbb{C})}(S_4)$, we break into two cases, when $b = 0$ or $b \neq 0$. If $b = 0$ then the matrices $C(a,0)$ are scalar and form the kernel of φ. If $b \neq 0$, then $\varphi(C(a,b)) = \varphi(C(a/b,1))$. For all $a \in \mathbb{C}\backslash\{1,-3\}$, the matrix $C(a/b,1)$ induces a nontrivial automorphism of $\mathbb{P}^3$ and thus does not lie in the kernel of φ. Thus by letting $\lambda = a/b$, we obtain that the normalizer of $S_4 < \mathrm{PGL}(4,\mathbb{C})$ is the subgroup stated in the proposition. □

This allows us to define the moduli of smooth symmetric cubic surfaces.

Proposition 2.11. The GIT quotient $\mathcal{S} = N_{\mathrm{PGL}(4,\mathbb{C})}(S_4)\backslash\!\backslash\mathbb{P}\mathcal{V}^{\mathrm{sm}}$ is the moduli space of smooth symmetric cubic surfaces.

Proof. Suppose that two symmetric cubic forms $f_1, f_2 \in \mathbb{P}\mathcal{V}^{\mathrm{sm}}$ determine isomorphic symmetric cubic surfaces $X = Z(f_1)$ and $Y = Z(f_2)$. Since such an isomorphism $\varphi : X \xrightarrow{\sim} Y$ of varieties preserves their respective canonical classes K_X and K_Y, the map extends to respect their anticanonical embeddings into $\mathbb{P}^3$. Thus such an isomorphism φ must be the restriction of a linear automorphism coming from the ambient projective space $\mathbb{P}^3$. Moreover, the automorphism groups of X and Y must be preserved under such an isomorphism, and so the S_4-action on X must be sent to the S_4-action on Y. Thus two symmetric cubic surfaces are projectively equivalent when their symmetric cubic forms differ by an element of the normalizer of $S_4 < \mathrm{PGL}(4,\mathbb{C})$, which was explicitly calculated in Proposition 2.10. □

2.4. **Symmetry and stability.** Having defined the moduli space of smooth symmetric cubic surfaces $\mathcal{S}$ in Proposition 2.11, we would like to define the analogous moduli spaces of stable and semistable symmetric cubic surfaces. In order to do this, we first must explore how (semi)stability interacts with symmetry.

Proposition 2.12. Let $f \in \mathcal{V}$ be a nonzero symmetric homogeneous form defining a semistable cubic surface. Then the singularities of $V(f)$ are either $4A_1$ or $3A_2$.

Proof. We first see that if $X = V(f)$, then the geometric S_4 action it inherits by symmetry is actually a subgroup of the automorphism group. This is clear if X is smooth, since the cubic surface is anticanonically embedded, but a small argument is needed if X isn't smooth. Suppose towards a contradiction that any non-trivial element $g \in S_4$ acted trivially on X. Then X would lie in the g-fixed subspace of $\mathbb{P}^3$, which is a hyperplane or intersection of hyperplanes, a contradiction.

Since a semistable cubic surface is normal by Serre's criterion [Gro65, 5.10], we can refer to the classification of automorphism groups of normal cubic surfaces due to Sakamaki [Sak10, Table 3]. It is clear that S_4 cannot be a subgroup of any of the automorphism groups except $4A_1$ where it is equality, and $3A_2$, where we make use of the semidirect product $K_4 \rtimes S_3 \cong S_4$. □

We now look to see if any such symmetric singular cubic surfaces do exist. One of the most famous singular cubic surfaces is symmetric:

Definition 2.13. The *Cayley nodal cubic surface*, defined by the elementary symmetric homogeneous form m_{111} is a singular cubic surface with four nodes. Its automorphism group is S_4, which permutes these four nodes [Sak10]. It is pictured in Figure 1.

Conveniently, the normalizer of S_4 in $\mathrm{PGL}(4, \mathbb{C})$ appears in the following proposition, which characterizes the Cayley cubic surface as the unique cubic surface with four nodes (c.f. [BW79]).

Proposition 2.14. Let $f \in \mathcal{W}$ be a nonzero form defining a cubic surface with four nodes. Then there exists a *unique* change of coordinates $g \in N_{\mathrm{PGL}(4,\mathbb{C})}(S_4)$ so that $g \cdot f$ is the Cayley nodal cubic.

Proof. Given any other cubic surface with four nodes, there is a projective change of coordinates turning it into the Cayley nodal cubic by sending the four nodes to the four nodes of the Cayley cubic. This change of coordinates is unique up to permutation of the nodes since $\mathrm{PGL}(4, \mathbb{C})$ is simply 4-transitive. However there is a unique automorphism of the Cayley nodal cubic corresponding to any permutation of the nodes, since its automorphism group is the symmetric group S_4. □

This has an immediate corollary to our study of stable symmetric cubic surfaces, which allows us to understand the moduli space.

Corollary 2.15. Any symmetric stable cubic surface which is not smooth lies in the $N_{\mathrm{PGL}(4,\mathbb{C})}(S_4)$ orbit of the Cayley nodal cubic surface.

Corollary 2.16. The moduli space of stable symmetric cubic surfaces

$$\mathcal{S}^s = N_{\mathrm{PGL}(4,\mathbb{C})}(S_4) \backslash\!\backslash \mathbb{P}(\mathcal{V}^s)$$

has the property that $\mathcal{S} \subseteq \mathcal{S}^s$, and $\mathcal{S}^s \smallsetminus \mathcal{S} = \{C\}$ is one point, which is the Cayley nodal cubic.

What about semistability? By Proposition 2.12 any semistable symmetric cubic which isn't stable must have three cusps, and we know there is a unique semistable non-stable cubic surface in the moduli space $\mathcal{M}^{ss}$ by Lemma 2.5. So it suffices to check if there is any projective change of coordinates exhibiting the tricuspidal cubic surface as a symmetric homogeneous form.

Computation 2.17. The tricuspidal cubic surface $z_0^3 - z_1z_2z_3$ is projectively equivalent to the homogeneous form $4m_{21} + 4m_{111}$ via the change of basis matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1. \end{pmatrix}$$

Why this works. Since S_4 acts faithfully on the cubic surface, it must map singularities to singularities, hence if a symmetric cubic surface has three cusps, they form an S_4-set. Viewing $\mathbb{P}^3$ as an S_4-space, we see there is a unique point in $\mathbb{P}^3$ with full isotropy S_4, and no points with isotropy A_4. Hence the three cusps must form a transitive S_4-set, isomorphic to S_4/D_8. We check that there is a unique such collection of three points in $\mathbb{P}^3$, namely $[1:1:-1:-1]$, $[1:-1:1:-1]$, and $[1:-1:-1:1]$. Since $\mathrm{PGL}(4,\mathbb{C})$ is 3-transitive, if we can the tricuspidal cubic above into a symmetric form, we must map its cusps $[0:1:0:0]$, $[0:0:1:0]$, and $[0:0:0:1]$ to the points forming the S_4/D_8 orbit above, hence the three rightmost columns in the matrix we found. A computation then forces the first column to consist of all 1's. □

Corollary 2.18. The moduli space of semistable symmetric cubic surfaces

$$\mathcal{S}^{ss} := N_{\mathrm{PGL}(4,\mathbb{C})}(S_4)\backslash\backslash\mathbb{P}(\mathcal{V}^{ss})$$

has exactly one point not in the stable moduli space, corresponding to the tricuspidal curve $m_{21} + m_{111}$.

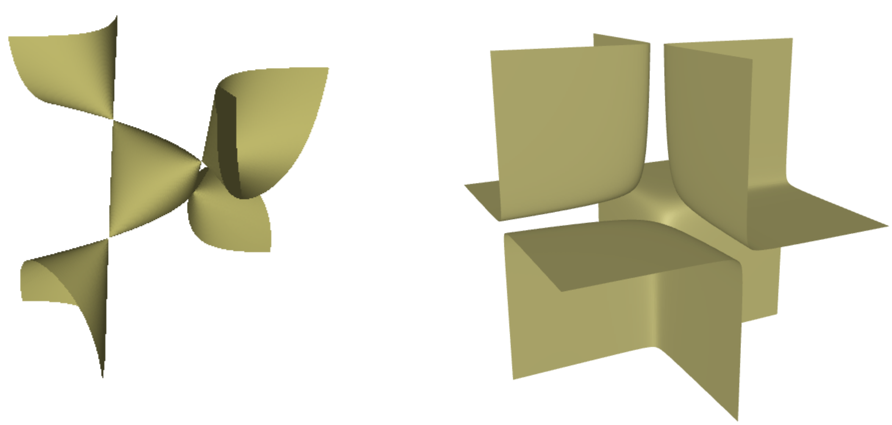


Figure 1. Left: the Cayley nodal cubic surface. Right: the tricuspidal cubic surface

The inclusion $\mathcal{V} \to \mathcal{W}$ of parameter spaces descends to a inclusion $\iota : \mathcal{S} \to \mathcal{M}$ of the moduli of symmetric cubic surfaces into the whole moduli space. The pullback $\iota^*\widehat{\mathcal{M}}$ is then the moduli space of smooth marked symmetric cubic surfaces. Then just as before, the pullback construction let's us conclude that $\iota^*\widehat{\mathcal{M}}$ is isomorphic to the moduli space of symmetric cubic surfaces equipped with a line $\widetilde{\mathcal{S}}$.

3. Reviewing Allcock–Carlson–Toledo

In this section, we outline the construction of the Allcock–Carlson–Toledo period map [ACT02]. Let $S = Z(f)$ denote a cubic surface in $\mathbb{P}^3$. Since $H^2(S, \mathbb{Z})$ admits a type $(1,1)$ Hodge structure, the natural period map is constant. In their seminal paper, Allcock–Carlson–Toledo showed that there is an weight 3 Hodge structure associated to S whose periods entirely capture its geometry. This is the Hodge structure of the cyclic cubic threefold T, realized as a degree 3 cover of $\mathbb{P}^3$ with branch locus S. In coordinates,

$$T = \{t^3 = f(z_0, z_1, z_2, z_3)\} \subset \mathbb{P}^4,$$

and the deck group $\langle \tau \rangle$ of the cover acts on T by multiplying the t-coordinate by the 3rd root of unity ω. The pair $(H^3(T, \mathbb{Z}), \tau)$ forms a so-called *Eisenstein Hodge structure*. The period domain for such Hodge structures is complex hyperbolic 4-space $\mathbb{CH}^4$. There is a natural period map from the moduli of smooth cubic surfaces $\mathcal{M}$ to an arithmetic quotient of $\mathbb{CH}^4$ by $P\hat{\Gamma} = \mathrm{PU}(4, 1, \mathbb{Z}[\omega])$:

$$\mathcal{P} : \mathcal{M} \to P\hat{\Gamma} \backslash \mathbb{CH}^4.$$

By the Riemann extension theorem, the map $\mathcal{P}$ extends uniquely to the stable cubic surfaces $\mathcal{M}^s$. The main theorem of [ACT02] is that $\mathcal{P}$ is a biholomorphism of analytic spaces $\mathcal{M}^s \cong P\hat{\Gamma} \backslash \mathbb{CH}^4$, and moreover it is an isomorphism of orbifolds. After reviewing their work, we will build an analogous uniformization of the moduli space of symmetric cubic surfaces.

3.1. **Basic cohomology knowledge.** Given a smooth cubic surface S defined by the cubic form f, we associate to it the cyclic cubic 3-fold $T := \{t^3 = f\}$ which defines a degree 3 branched covered $\mathbb{P}^3$ over S. The Lefschetz hyperplane theorem and Poincaré duality tells us that T and $\mathbb{P}^3$ have the same cohomology away from the middle degree 3, with Hodge numbers equal to $h^{i,i}(T) = 1$ for $i = 0, 1, 2, 3$. An Euler characteristic calculation tells us that $H^3(T, \mathbb{Z})$ is rank 10, and since there are no holomorphic 3-forms on T, i.e. $h^{3,0}(T) = h^{0,3}(T) = 0$, the middle Hodge numbers of T are $h^{2,1} = h^{1,2} = 5$.

3.2. **The module structure on cohomology.** The deck group $\langle \tau \rangle$ of the triple branched cover $T \to \mathbb{P}^3$ acts on T by multiplying the t-coordinate by a 3rd root of unity ω. Since fixed vectors of the induced action on cohomology $H^3(T, \mathbb{Z})$ must come from $H^3(\mathbb{P}^3, \mathbb{Z}) = 0$ via transfer, τ acts on the (real) cohomology of T without fixed points. Thus the minimal polynomial of the τ-action on $H^3(T)$ is $z^2 + z + 1$, and $H^3(T, \mathbb{Z})$ inherits the structure of a free $\mathbb{Z}[\omega]$-module of dimension five.

The Hodge decomposition on $H^3(T, \mathbb{C})$ forms a direct sum decomposition

$$H^3(T, \mathbb{C}) = H^{2,1}(T) \oplus H^{1,2}(T),$$

where the two summands are isomorphic dimension five vector spaces and are exchanged by complex conjugation. Since τ acts holomorphically on T, ω acts on $H^3(T, \mathbb{Z})$ as a real operator, and so it preserves the Hodge decomposition. Thus the eigenspace decomposition $H^3(T) = H^3_\omega(T) \oplus H^3_{\bar{\omega}}(T)$ is compatible with the Hodge decomposition. Selecting the $\bar{\omega}$-summand, we get a Hodge-eigenspace direct sum decomposition

$$H^3_{\bar{\omega}}(T) = H^{2,1}_{\bar{\omega}}(T) \oplus H^{1,2}_{\bar{\omega}}(T).$$

There is a naturally associated Hermitian $\mathbb{Z}[\omega]$-valued form h on $H^3(T,\mathbb{Z})$ coming from the cup product $\langle\cdot,\cdot\rangle$:

$$h(a,b) := \frac{\langle(\tau-\tau^{-1})a,b\rangle - (\omega-\omega^{-1})\langle a,b\rangle}{2}.$$

With respect to this form h, the Hermitian pair $(H^3_{\bar\omega}(T),h)$ is a signature $(4,1)$ complex inner product space. A key point in [ACT02] is that this direct sum decomposition is orthogonal with respect to h, $H^{2,1}_{\bar\omega}(T)$ is 1-dimensional and negative-definite with respect to h, and that $H^{1,2}_{\bar\omega}(T)$ is 4-dimensional and positive-definite with respect to h.

3.3. **Moduli of framed cubic surfaces.** We have already defined markings and constructed the moduli space of marked cubic surfaces. A related but slightly different idea must be studied, which is the notion of a *framing*; this is a marking of the cohomology of the cyclic cubic threefold T associated to a cubic surface S.

Definition 3.1. A *framing* of the cubic surface S is an isometry ψ from the $\mathbb{Z}[\omega]$-lattice $(H^3(T,\mathbb{Z}),h)$ to the abstract indefinite $\mathbb{Z}[\omega]$-lattice $\Lambda=\mathbb{Z}[\omega]^{4,1}$ (recall that Λ is unique up to isometry [All00]). Two framings (S_1,ψ_1) and (S_2,ψ_2) are equivalent if there exists some $g\in\mathrm{PGL}(4,\mathbb{C})$ such that $\psi_1=\tilde{g}^*\psi_2$. The space of equivalence classes of framed cubic surfaces (S,ψ) is denoted by $\mathcal{F}$.

Allcock–Carlson–Toledo showed that $\mathcal{F}$ is a complex manifold. There is a natural action of the arithmetic group $\hat\Gamma=\mathrm{U}(4,1,\mathbb{Z}[\omega])$ on the space of framed cubic surfaces by $\gamma\cdot(S,\psi)=(S,\psi\circ\gamma^{-1})$.

3.4. **The period mapping.** The space of negative lines in the space $\mathbb{C}^{4,1}=\Lambda\otimes_{\mathbb{Z}[\omega]}\mathbb{C}$ is the complex hyperbolic 4-space $\mathbb{CH}^4$. We can now define the period mapping of framed cubic surfaces $\widetilde{\mathcal{P}}:\mathcal{F}\to\mathbb{CH}^4$ to be

$$\widetilde{\mathcal{P}}:(S,\psi)\longmapsto\mathbb{P}(\psi(H^{2,1}_{\bar\omega}(T)))\in\mathbb{CH}^4\subset\mathbb{CP}^4.$$

The main theorem of [ACT02] is that this map is an open embedding, and descends through the $\hat\Gamma$-quotient to an isomorphism $\mathcal{P}$ of moduli spaces $\mathcal{M}\cong P\hat\Gamma\backslash(\mathbb{CH}^4-\mathcal{H})$, where $\mathcal{H}$ is the locally finite hyperplane arrangement determined by reflections over short roots $\delta\in\Lambda$:

$$\mathcal{H}=\bigcup_{h(\delta,\delta)=1}\mathrm{Fix}(\mathrm{Ref}_\delta).$$

This isomorphism extends to the moduli space of stable cubic surfaces $\mathcal{M}^s\cong P\hat\Gamma\backslash\mathbb{CH}^4$. They remark that this map extends to an analytic isomorphism of Deligne–Mumford stacks; for more details on this stacky structure, see the papers of Kudla–Rapoport [KR12] and Zheng [Zhe21].

Remark 3.2. Note that originally Allcock–Carlson–Toledo used the negative-definite line $H^{1,2}_{\omega}(T)$ to define the period data of cubic surfaces, but this makes the period mapping anti-holomorphic, as pointed out by Beauville [Bea09]. This is why we adopted the $\bar\omega$ convention instead.

Finally, consider the group homomorphism $\mathbb{Z}[\omega]\to\mathbb{F}_3$ which sends ω to 1. Then $\mathbb{Z}[\omega]^{4,1}\otimes_{\mathbb{Z}[\omega]}\mathbb{F}_3\cong\mathbb{F}_3^{4,1}$, which induces a homomorphism

$$\varphi:\hat\Gamma\to\mathrm{PO}(4,1,\mathbb{F}_3).$$

Allcock–Carlson–Toledo showed that this homomorphism is surjective, with kernel denoted by $\hat{\Gamma}'$. The following is well-known which we include for the sake of completeness (see [ACT02, Section 2.12] and [CCN+85, pg. 26]):

Proposition 3.3. There is an exceptional isomorphism of finite groups $W(E_6) \cong \mathrm{PO}(4, 1, \mathbb{F}_3)$.

Proof sketch. The Weyl group $W(E_6)$ acts on the 6-dimensional root lattice E_6. This has index 3 in the weight lattice. The root lattice modulo 3 times the weight lattice is 5-dimensional vector space over $\mathbb{F}_3$. This space inherits an inner product q from the root lattice by reducing mod 3 the inner product of lattice vectors. Thus every element of $W(E_6)$ descends to an automorphism of $\mathbb{F}_3^5$ which preserves this non-degenerate symmetric bilinear form q. This yields the homomorphism $W(E_6) \to \mathrm{O}(q, \mathbb{F}_3)$. Post-composing with the projectivization, we obtain the group homomorphism

$$W(E_6) \to \mathrm{PO}(q, \mathbb{F}_3).$$

Any two non-degenerate symmetric bilinear forms q and q' are equivalent over $\mathbb{F}_3$ [MH73], and so the group $\mathrm{PO}(q, \mathbb{F}_3)$ and $\mathrm{PO}(4, 1, \mathbb{F}_3)$ isomorphic. One can then calculate that $|W(E_6)| = |\mathrm{PO}(q, \mathbb{F}_3)| = 51840$. By almost simplicity of $W(E_6)$ and non-triviality of this homomorphism, this map is an isomorphism. $\square$

By Proposition 3.3, we have that $\hat{\Gamma}/\hat{\Gamma}' \cong W(E_6)$, and so the cover of $\mathcal{M}$ that $\hat{\Gamma}' < \hat{\Gamma}$ corresponds to is the space of cubic surfaces equipped with a line $\widetilde{\mathcal{M}}$. Once we have analogously uniformized moduli space the symmetric cubic surfaces, a special symmetric subgroup of $\mathrm{PO}(4, 1, \mathbb{F}_3)$ will be determined that will contain the monodromy group of the cover $\widetilde{\mathcal{S}} \to \mathcal{S}$.

4. Uniformization of the symmetric moduli space

As was reviewed in the previous section, Allcock–Carlson–Toledo showed that the moduli space of stable cubic surfaces $\mathcal{M}^s$ admits the structure of a complex ball quotient. Specializing to the stable symmetric locus $\mathcal{S}^s$, we will study this space through Hodge theory and analogously realize it as a ball quotient. Although their stated goals and some of the technology used are different, we take inspiration from and owe an intellectual debt to the work of Yu–Zheng [YZ20].

To understand the moduli space of symmetric cubic surfaces, we need to understand how S_4 acts on the period data. More specifically, we want to understand the $\bar{\omega}$-eigenspace in cohomology $H^3_{\bar{\omega}}(T)$ as an S_4-representation. Any S_4-equivariant automorphism of the cubic surface S will lift to give a nontrivial S_4-equivariant action on T, thus the cohomology group $H^3_{\bar{\omega}}(T)$ while preserving the Hodge decomposition. To understand this action, we will express the periods of a given symmetric cubic surface in terms of differential forms.

4.1. **Residue calculus and symmetric cubic 3-folds.**

Griffiths' theory of residues [Gri69] will help us make explicit the action of S_4 on the invariant cohomology of a cyclic cubic 3-fold. His foundational work on rational integrals allows us to assert the following:

Proposition 4.1. Let $S = Z(f)$ be a cubic surface, $f_S = t^3 - f$ the cubic form defining its associated cyclic cubic 3-fold $T = Z(f_S)$, and Ω the standard volume form on $\mathbb{P}^4$. The map

$$\mathbb{C}[z_0, z_1, z_2, z_3, t]_{(1)} \longrightarrow H^{2,1}(T)$$
$$P \longmapsto \mathrm{Res}_T\left(\frac{P\Omega}{f_S^2}\right)$$

is an isomorphism of vector spaces, under which the line $\mathbb{C}\langle t\rangle$ maps to $H^{2,1}_{\bar{\omega}}(T)$.

If we additionally assume that f defined an symmetric cubic surface, the cyclic cubic 3-fold T also admits an S_4 symmetry. Thus the cohomology of T inherits the structure of an S_4-representation, which we seek to determine.

Lemma 4.2. For any symmetric cyclic cubic 3-fold T, the cohomology group $H^{2,1}(T)$ is isomorphic as an S_4-representation to $\mathbb{C} \oplus V$, where V is the standard S_4-permutation representation on $\mathbb{C}^4$.

Proof. The meromorphic differential forms

$$\left\langle \frac{z_0\Omega}{f_S^2}, \frac{z_1\Omega}{f_S^2}, \frac{z_2\Omega}{f_S^2}, \frac{z_3\Omega}{f_S^2}, \frac{t\Omega}{f_S^2} \right\rangle$$

give us a basis for $H^{2,1}(T)$. From this we can explicitly compute the induced structure on $H^{2,1}(T)$ as an S_4-representation. Recall that S_4 acts by linear permutation automorphisms on $z_0, \ldots, z_3$ and acts trivially t. Moreover, $\sigma^*\Omega = \Omega$ and $\sigma^* f_S = f_S$ for all $\sigma \in S_4$ since the symmetric group leaves invariant the cubic form f_S and the volume form Ω. The claim follows. □

4.2. **Equivariant framings and the local period map.** For every cyclic cubic threefold T, we have that

$$H^3_{\bar{\omega}}(T) \cap H^{1,2}(T) = H^{1,2}_{\bar{\omega}}(T)$$

is a positive hyperplane in the signature $(4,1)$-space $H^3_{\bar{\omega}}(T)$. Since S_4 acts on T by holomorphic automorphisms and commutes with the deck group $\langle\tau\rangle$, the induced action on $H^3_{\bar{\omega}}(T)$ must act trivially on the line $H^{2,1}_{\bar{\omega}}(T)$ defining the period data. After complex conjugating, Lemma 4.2 tells us that $H^{1,2}_{\bar{\omega}}(T)$ is isomorphic, as an S_4-representation, to the standard permutation representation. The hyperplane $H^{1,2}_{\bar{\omega}}(T)$ is then uniquely determined as an S_4-representation by the 1-dimensional trivial S_4-representation $\mathbb{C} \subset H^{1,2}_{\bar{\omega}}(T)$.

The ambient signature $(4,1)$-space $\mathbb{C}^{4,1} \cong H^3_{\bar{\omega}}(T)$ is where periods of cubic surfaces S live in. To refine our period data equivariantly, we will use the fixed locus $H^3_{\omega}(T)_1$ of the S_4-action on $H^3_{\omega}(T)$ to define the period domain of symmetric cubic surfaces. By the above discussion, $H^3_{\omega}(T)_1$ is a signature $(1,1)$ complex inner product space.

Let T be a S_4-invariant cyclic cubic threefold associated to the symmetric cubic surface S, and let $\sigma_T : S_4 \times H^3(T,\mathbb{Z}) \to H^3(T,\mathbb{Z})$ be the S_4-action induced on the $\mathbb{Z}[\omega]$-module $H^3(T,\mathbb{Z})$. Set Λ equal to the unique $\mathbb{Z}[\omega]$-lattice of signature $(4,1)$ abstractly isomorphic to $H^3(T,\mathbb{Z})$, and σ an S_4-action on Λ abstractly isomorphic to the action σ_T on $H^3(T,\mathbb{Z})$.

Definition 4.3. An *equivariant framing* is a pair (S, λ) of a symmetric cubic surface S and a framing

$$\lambda : (H^3(T, \mathbb{Z}), \sigma_T) \xrightarrow{\sim} (\Lambda, \sigma)$$

which sends the action σ_T on $H^3(T, \mathbb{Z})$ to the action σ on Λ. Two equivariant framings (S_1, λ_1) and (S_2, λ_2) are equivalent if there exists some $g \in N_{\mathrm{PGL}(4,\mathbb{C})}(S_4)$ such that $\lambda_1 = \tilde{g}^* \lambda_2$. Let $\mathcal{G}$ denote the space of equivalence classes of equivariantly framed symmetric cubic surfaces (S, λ).

Proposition 4.4 ([YZ20, Proposition 4.2])**.** The space $\mathcal{G}$ is a complex manifold.

Let $\Lambda_{\mathbb{C},1} \subset \Lambda_{\mathbb{C}} = \Lambda \otimes_{\mathbb{Z}[\omega]} \mathbb{C}$ denote the fixed locus of the S_4-action σ on $\Lambda_{\mathbb{C}}$.

Definition 4.5. The *symmetric period domain* $\mathbb{D}$ associated to the moduli of equivariantly marked symmetric cubic surfaces $\mathcal{G}$ is the Hermitian symmetric domain

$$\mathbb{D} = \mathbb{P}\{x \in \Lambda_{\mathbb{C},1} \cong \mathbb{C}^{1,1} : h(x, \overline{x}) < 0\}.$$

Clearly $\mathbb{D} \cong \mathbb{CH}^1$, the complex hyperbolic line. Equivalently, $\mathbb{D}$ is the real hyperbolic plane.

The following diagram contains most of the spaces of interest. The main content of this section will be showing injectivity of top left horizontal map $\mathcal{G} \to \mathcal{F}$. The right column of horizontal period maps are injective by [ACT02]. The remaining horizontal maps in the left column are injective by definition and the pullback construction.

$$\begin{array}{lccccc}
\text{framed moduli} & \mathcal{G} & \longrightarrow & \mathcal{F} & \longrightarrow & \mathbb{CH}^4 \\
 & \downarrow & & \downarrow & & \downarrow \\
\text{marked moduli} & \widetilde{\mathcal{S}} & \longrightarrow & \widetilde{\mathcal{M}} & \longrightarrow & \hat{\Gamma}' \backslash \mathbb{CH}^4 \\
 & \downarrow & & \downarrow & & \downarrow \\
\text{moduli} & \mathcal{S} & \longrightarrow & \mathcal{M} & \longrightarrow & \hat{\Gamma} \backslash \mathbb{CH}^4
\end{array}$$

Proposition 4.6. Let $\mathcal{H}^{S_4} = \mathbb{D} \cap \mathcal{H}$ denote the symmetric discriminant locus in the period domain. The natural map $\mathcal{G} \to \mathcal{F}$ is injective. Thus the local symmetric framed period mapping $\widetilde{\mathcal{P}} : \mathcal{G} \to \mathbb{D}$ injective, and moreover is an open embedding onto its image $\mathbb{D} - \mathcal{H}^{S_4}$. Moreover, its extension to the stable locus $\mathcal{G}^s$ is surjective.

Proof. Suppose we are given two equivariantly framed symmetric cubic surfaces $(S_1, \lambda_1), (S_2, \lambda_2) \in \mathcal{G}$ that map to the same point in $\mathcal{F}$. Then there exists an linear isomorphism of varieties $g : S_1 \xrightarrow{\sim} S_2$ along with a unique up to deck transformations lift $\tilde{g} : T_1 \xrightarrow{\sim} T_2$ satisfying

$$\tilde{g}^* = \lambda_1^{-1} \circ \lambda_2 : H^3(T_2, \mathbb{Z}) \to H^3(T_1, \mathbb{Z}),$$

which is an isometry of $\mathbb{Z}[\omega]$-lattices. Since λ_1 and λ_2 are compatible with the S_4-action, so is $\tilde{g}^*$. By [Zhe21, Theorem 1.1], the equivariantly framed cubic surfaces (S_1, λ_1) and (S_2, λ_2) represent the same point in $\mathcal{G}$. This proves injectivity of the map $\mathcal{G} \to \mathcal{F}$. Commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{G} & \hookrightarrow & \mathcal{F} \\
\downarrow & & \downarrow \\
\mathbb{D} & \hookrightarrow & \mathbb{CH}^4
\end{array}$$

then implies that the local symmetric framed period map $\mathcal{G} \to \mathbb{D}$ is injective. Moreover, since the derivative of the period map $\widetilde{\mathcal{P}} : \mathcal{F} \to \mathbb{CH}^4$ is injective everywhere, so is the derivative of $\widetilde{\mathcal{P}} : \mathcal{G} \to \mathbb{D}$. Thus the local symmetric framed period map induces a diffeomorphism onto its image $\mathbb{D} - (\mathbb{D} \cap \mathcal{H}) = \mathbb{D} - \mathcal{H}^{S_4}$. Since the local period map on the stable framed moduli space $\mathcal{F}^s$ has image $\mathbb{CH}^4$ [ACT02, Theorem 3.17], the image of the local symmetric period map $\mathcal{G}^s \to \mathbb{D}$ is surjective, thereby proving the claim. □

4.3. **The global period map.** Since S_4 acts on any symmetric cubic surface and its associated cyclic cubic 3-fold T, it embeds into the arithmetic group $\mathrm{U}(4, 1, \mathbb{Z}[\omega])$ via its action on $H^3(T, \mathbb{Z})$ (note that this map is injective, since S_4 embeds into the mod 3 reduction $\mathrm{PO}(4, 1, \mathbb{F}_3)$ via its action on the 27 lines). To study how the local period map descends to yield a uniformization of the symmetric moduli space by the global period map, we must determine the normalizer of S_4 in a few groups of interest.

Proposition 4.7. There is an isomorphism of groups $N_{\mathrm{U}(4,1)}(S_4) \cong \mathrm{U}(1, 1) \times (\mathrm{U}(1) \times S_4)$.

Proof. Recall that S_4 acts on $\mathbb{C}^{4,1}$ by the standard permutation representation on the positive 4-space and the negative 1-space. This splits the space into a direct sum of irreducible representations $W \oplus \mathbb{C} \oplus \mathbb{C}$, where W denotes the irreducible S_4-representation of dimension 3, and the sum of the two trivial representations form a signature $(1, 1)$-space. By Proposition 2.9, it suffices to determine what the centralizer of S_4 is within $\mathrm{U}(4, 1)$. By centrality, we can deal with the 3-dimensional factor and $(1, 1)$-factor individually.

Schur's lemma tells us that the S_4-centralizer acts by scalars on W, and thus is isomorphic to a copy of $\mathrm{U}(1)$ acting on W. On the signature $(1, 1)$-factor, the S_4-action is trivial, and thus every element of $\mathrm{U}(1, 1)$ arises at an automorphism of the representation $\mathbb{C} \oplus \mathbb{C}$. Thus the normalizer is the product of the normalizers on each factor, proving that $N_{\mathrm{U}(4,1)}(S_4) \cong \mathrm{U}(1, 1) \times (\mathrm{U}(1) \times S_4)$. □

Proposition 4.8. There is an isomorphism of groups $N_{\mathrm{U}(4)}(S_4) \cong \mathrm{U}(1) \times S_4$, where $\mathrm{U}(1)$ acts on the permutation representation V by scalars.

Proof. Using Proposition 2.9, we need only determine the centralizer of S_4 to generate the normalizer. Yet again, these are the unitary scalar matrices. □

Proposition 4.9. The group $\Gamma = N_{\mathrm{U}(4,1,\mathbb{Z}[\omega])}(S_4)$ is naturally an arithmetic subgroup of $N_{\mathrm{U}(4,1)}(S_4)$. Moreover, we have an isomorphism $\Gamma \cong \mathrm{Aut}(\mathrm{diag}(4, -1), \mathbb{Z}[\omega]) \times (\langle -\omega \rangle \times S_4)$

Proof. For arithmeticity, see [YZ20, Appendix A]. As before, Proposition 2.9 tells us that it suffices to determine the centralizer in $\mathrm{U}(4, 1, \mathbb{Z}[\omega])$, which we shall do on each factor of the S_4-representation. The Eisenstein lattice $\mathbb{Z}[\omega]^5 \subset \mathbb{C}^{4,1}$ intersects the S_4-representations W and $\mathbb{C} \oplus \mathbb{C}$ in rank 2 and 3 Eisenstein lattices, respectively. The centralizer is then a subgroup of the product of the automorphism group of these lattices. These symmetries must be automorphisms of $\mathbb{C}^{4,1}$ which preserve the whole Eisenstein lattice.

By Proposition 4.7, the S_4-centralizer of the rank 3 sublattice must be scalars $\theta \in \mathrm{U}(1)$ which preserves $\mathbb{Z}[\omega]$; this subgroup is $\langle -\omega \rangle$. A standard calculation tells us that generators for this rank

2 sublattice are given by $(1,1,1,1,0)$ and $(0,0,0,0,1)$, thus the signature $(4,1)$ form restricts to the bilinear form $\mathrm{diag}(4,-1)$. Since S_4 acts trivially on this rank 2 sublattice, the full automorphism group $\mathrm{Aut}(\mathrm{diag}(4,-1), \mathbb{Z}[\omega])$ which preserves the Eisenstein lattice constitutes the centralizer on this factor. This proves the claim. □

From these calculations, we can conclude the following which is an application of a well-known fact about locally symmetric varieties [YZ20, Proposition A.1]:

Proposition 4.10. Let $\hat{G} = \mathrm{U}(4,1)$, $\hat{K} = \mathrm{U}(4)$, and $\hat{\Gamma} = \mathrm{U}(4,1,\mathbb{Z}[\omega])$. Set $G = N_{\hat{G}}(S_4)$, $K = N_{\hat{K}}(S_4)$, and $\Gamma = N_{\hat{\Gamma}}(S_4)$. The holomorphic embedding $G/K \to \hat{G}/\hat{K}$ descends to a generically injective finite normalization

$$\Gamma \backslash G/K \to \hat{\Gamma} \backslash \hat{G}/\hat{K}.$$

Following [YZ20, Proposition 4.10], we can prove this more precise version of Theorem 1.1:

Theorem 4.11. The local period map $\widetilde{\mathcal{P}} : \mathcal{G} \to \mathbb{D}$ descends to isomorphisms $\mathcal{S} \cong P\Gamma \backslash (\mathbb{D} - \mathcal{H}^{S_4})$ and $\mathcal{S}^s \cong P\Gamma \backslash \mathbb{D}$. Moreover, this is an isomorphism of analytic orbifolds compatible with the uniformization of the moduli of cubic surfaces, so the totally geodesic embedding

$$P\Gamma \backslash \mathbb{D} \to P\hat{\Gamma} \backslash \mathbb{CH}^4$$

is a modular embedding of locally symmetric orbifolds. This map compatibly extends to an isomorphism of the semistable symmetric moduli space $\mathcal{S}^{ss} \cong \overline{P\Gamma \backslash \mathbb{D}}$, where the latter space denotes the Satake compactification of the arithmetic quotient.

Proof. We will first show that the map $\widetilde{\mathcal{P}}$ descends to a well-defined map $\mathcal{P}([S]) = [\mathbb{P}(H^{2,1}_{\bar{\omega}}(T))]$. Let $f_1, f_2 \in \mathcal{V}^{\mathrm{sm}}$ be two smooth symmetric cubic forms with equivariant framings λ_1, λ_2 of their associated cubic 3-folds T_1, T_2. Suppose there is some $g \in N_{\mathrm{PGL}(4,\mathbb{C})}(S_4)$ such that $g(f_1) = f_2$. This induces the $\mathbb{Z}[\omega]$-isometry

$$\tilde{g}^* : H^3(T_2, \mathbb{Z}) \to H^3(T_1, \mathbb{Z}).$$

We will show that $\gamma = \lambda_1 \circ \tilde{g}^* \circ \lambda_2^{-1} \in \Gamma$. Since $g \in N_{\mathrm{PGL}(4,\mathbb{C})}(S_4)$, $g\sigma g^{-1} = \sigma' \in S_4$, thus we have a commutative diagram

$$\begin{array}{ccccccc} \Lambda & \xrightarrow{\lambda_2^{-1}} & H^3(T_2,\mathbb{Z}) & \xrightarrow{\tilde{g}^*} & H^3(T_1,\mathbb{Z}) & \xrightarrow{\lambda_1} & \Lambda \\ \downarrow{\scriptstyle \sigma'} & & \downarrow{\scriptstyle \sigma'^*} & & \downarrow{\scriptstyle \sigma*} & & \downarrow{\scriptstyle \sigma} \\ \Lambda & \xrightarrow[\lambda_2^{-1}]{} & H^3(T_2,\mathbb{Z}) & \xrightarrow[\tilde{g}^*]{} & H^3(T_1,\mathbb{Z}) & \xrightarrow[\lambda_1]{} & \Lambda \end{array}$$

Thus, as automorphisms of Λ, $\sigma' = \gamma^{-1}\sigma\gamma$, proving that $\gamma \in \Gamma$. This proves the map $\mathcal{P}$ is a well-defined and yields a commutative diagram

$$\begin{array}{ccc} \mathcal{G} & \xrightarrow{\widetilde{\mathcal{P}}} & \mathbb{D} \\ \downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow[\mathcal{P}]{} & P\Gamma \backslash \mathbb{D} \end{array}$$

The Riemann extension theorem tells us that $\mathcal{P}$ extends uniquely to the stable locus $\mathcal{S}^s$. By commutativity of this diagram and Proposition 4.6, the global period map $\mathcal{P} : \mathcal{S} \to P\Gamma\backslash(\mathbb{D} - \mathcal{H}^{S_4})$ and its stable extension $\mathcal{S}^s \to P\Gamma\backslash\mathbb{D}$ are surjective. We shall now show that $\mathcal{P}$ is injective.

Let $(S_1, \lambda_1), (S_2, \lambda_2) \in \mathcal{G}$ be two equivariantly framed symmetric cubic surfaces with associated cubic forms f_1, f_2, and suppose their periods represent the same point in $P\Gamma\backslash\mathbb{D}$. Then there exists some $\gamma \in \Gamma$ such that $\gamma \cdot \lambda_1(H^{2,1}_{\bar{\omega}}(T_1)) = \lambda_2(H^{2,1}_{\bar{\omega}}(T_2))$. Thus the map $\lambda_2^{-1} \circ \gamma \circ \lambda_1 : H^3(T_1, \mathbb{Z}) \to H^3(T_2, \mathbb{Z})$ preserved preserves the Eisenstein Hodge structures. By [Zhe21, Theorem 1.1], there exists some $g \in \mathrm{PGL}(4, \mathbb{C})$ such that $g(f_2) = f_1$ and $\tilde{g}^* = \lambda_2^{-1} \circ \gamma \circ \lambda_1$. To prove injectivity of $\mathcal{P}$, we want to show that $g \in N_{\mathrm{PGL}(4,\mathbb{C})}(S_4)$.

For any $\sigma \in S_4$ acting on $S_1 = Z(f_1)$, we have that $g^{-1}\sigma g$ acts on $S_2 = Z(f_2)$, which induces the following on the cohomology of the associated cyclic cubic 3-folds:

$$(\tilde{g}^{-1}\sigma\tilde{g})^* = \tilde{g}^*\sigma^*(\tilde{g}^{-1})^* = (\lambda_2^{-1}\gamma\lambda_1)(\lambda_1^{-1}\sigma^*\lambda_1)(\lambda_1^{-1}\gamma^{-1}\lambda_2) = \lambda_2^{-1}\gamma\sigma^*\gamma^{-1}\lambda_2.$$

Since $\gamma \in \Gamma$, we have that $\gamma\sigma^*\gamma^{-1} \in S_4$ as an automorphism of cohomology. Again by [Zhe21, Theorem 1.1], we have that $g\sigma g^{-1} \in S_4$, proving that $g \in N_{\mathrm{PGL}(4,\mathbb{C})}(S_4)$. Modular compatibility of the totally geodesic embedding is a consequence of Proposition 4.10. This proves the global period map satisfies the claimed properties.

By [ACT02, Theorem 8.2], the period map extends to the semistable locus for the total moduli space $\mathcal{M}^{ss} \to \overline{P\hat{\Gamma}\backslash\mathbb{CH}^4}$ and sends the unique semistable non-stable point to the unique boundary point of the Satake compactification. Since the embedding of locally symmetric orbifolds $P\Gamma\backslash\mathbb{D} \to P\hat{\Gamma}\backslash\mathbb{CH}^4$ is modular, the extension to their Satake compactifications is modular, and thus the tricuspidal point on $\mathcal{S}^{ss}$ maps to the unique boundary point of $\overline{P\Gamma\backslash\mathbb{D}}$, as claimed. □

Now that we have successfully uniformized the moduli space of symmetric cubic surfaces, we will begin our study of the monodromy group associated to the cover $\widetilde{\mathcal{S}} \to \mathcal{S}$, where $\widetilde{S}$ is the moduli of symmetric cubic surfaces equipped with a line.

Proposition 4.12. Consider the group homomorphism $\mathrm{U}(4, 1, \mathbb{Z}[\omega]) \to \mathrm{PO}(4, 1, \mathbb{F}_3)$ induced by the map $\omega \mapsto 1$. The subgroup corresponding to $\Gamma = N_{\mathrm{U}(4,1,\mathbb{Z}[\omega])}(S_4)$ has image $K_4 \times S_4$ in $\mathrm{PO}(4, 1, \mathbb{F}_3)$.

Proof. We appeal to the isomorphism explicitly traced out in the proof of Proposition 4.9, and determine the mod 3 reduction on each factor. As previously discussed, the S_4 factor survives the quotient by its action on the 27 lines. Since $\omega \mapsto 1$, the $\langle -\omega \rangle$ factor which acted on the rank 3 lattice by scaling is sent to $\langle -1 \rangle \cong C_2$. Finally, since the quadratic form $\mathrm{diag}(4, 1)$ reduces mod 3 to the quadratic form $\mathrm{diag}(1, -1)$, one can calculate that the group $\mathrm{Aut}(\mathrm{diag}(4, -1), \mathbb{Z}[\omega])$ has image isomorphic to $\mathrm{PO}(1, 1, \mathbb{F}_3) \cong C_2$. Thus we've shown the image of Γ is isomorphic to $K_4 \times S_4$. □

One may be tempted to conclude that the monodromy group of the cover $\widetilde{\mathcal{S}} \to \mathcal{S}$ is $K_4 \times S_4$. Indeed, [ACT10, Section 8] outlines why the monodromy groups associated to connected components of moduli of real cubic surfaces are the image of their fundamental groups in $\mathrm{PO}(4, 1, \mathbb{F}_3)$. However, all that Proposition 4.12 guarantees is that the monodromy group is contained in $K_4 \times S_4$. Remarkably, the philosophy of "big monodromy" fails to pin down our desired Galois group from purely Hodge theoretic considerations — it will be a proper subgroup of $K_4 \times S_4$! Further analysis using equivariant line geometry on cubic surfaces is required, which we carry out in the next section.

5. Calculating the monodromy group

In this section, we will determine the monodromy group of the cover $\widetilde{\mathcal{S}} \to \mathcal{S}$ by a combination of classical, moduli-theoretic, and computational techniques. We begin with the following basic fact:

Proposition 5.1. The automorphism group of a cubic surface acts faithfully on its lines.

Proof. Any automorphism φ of a cubic surface S preserves the canonical class K_S, and thus extends to the anticanonical embedding of S into $\mathbb{P}^3$, so $\varphi \in \mathrm{PGL}(4, \mathbb{C})$. Thus φ sends lines to lines on S, and any such φ which fixes all 27 lines must be the identity. □

Example 5.2. For symmetric cubic surfaces, this implies that $S_4 \subseteq W(E_6)$. A priori for different symmetric cubic surfaces we might obtain different conjugacy classes of S_4 in $W(E_6)$, however the connectivity of the moduli space of symmetric cubic surfaces guarantees this cannot occur. Thus when we discuss S_4 as a subgroup of $W(E_6)$ we are implicitly referring to this specific conjugacy class of subgroups.

Proposition 5.3. The symmetric monodromy group is a subgroup of $N_{W(E_6)}(S_4) \cong K_4 \times S_4$.

Proof. This is immediate by translating Proposition 4.12 along the exceptional isomorphism $W(E_6) \cong \mathrm{PO}(4, 1, \mathbb{F}_3)$. It can also be proved by leveraging Luna's étale slice theorem (c.f. [Lun73, PV94]) to argue there exists a universal deformation space for S_4-symmetric cubic surfaces, and therefore via descent, monodromy in the symmetric locus preserves the fiberwise S_4-action on a universal family of symmetric cubic surfaces, thereby normalizing S_4 in the full monodromy group $W(E_6)$.

The splitting of the short exact sequence

$$0 \to S_4 \to N_{W(E_6)}(S_4) \to K_4 \to 0. \tag{4}$$

claimed in the proposition is a computer verifiable computation. □

5.1. **Stability of the D_8 tritangent.** We can further restrict the symmetric monodromy via an understanding of how S_4 acts on the 27 lines on a symmetric cubic surface. The following result of the first named author proves that it is independent of the choice of smooth symmetric cubic surface.

Theorem 5.5. [Bra24, Theorem 1.2] On any smooth symmetric cubic surface, the 27 lines have orbits

$$[S_4/C_2^o] + [S_4/C_2^e] + [S_4/D_8]\,,$$

where $C_2^o = (1\ 2)$ is an odd copy of the cyclic group of order two, and $C_2^e = (1\ 2)(3\ 4)$ is an even copy of the cyclic group of order two.

Example 5.6. The *Fermat cubic surface* is defined by the symmetric homogeneous form m_3. Its 27 lines, with explicit labels and parametric equations, are given in the appendix of this paper

(Data A.1). The lines $\ell_1, \ldots, \ell_{12}$ lie in the S_4/C_2^o orbit, the lines $\ell_{13}, \ldots, \ell_{24}$ lie in the S_4/C_2^e orbit, and the lines $\ell_{25}, \ell_{26}, \ell_{27}$ form a tritangent which is the S_4/D_8 orbit. We refer to these three lines as the D_8-*tritangent*.

Remark 5.7. Once labels are fixed on the 27 lines, we can construct $W(E_6)$ as a permutation group, given as the adjacency-preserving permutations of the 27 lines. As a subgroup of S_{27} with the labeling of the lines coming from the Fermat cubic surface, the generators for W are listed in Data A.2.

Proposition 5.8. The three lines $\{\ell_{25}, \ell_{26}, \ell_{27}\}$ lie on every symmetric cubic surface, forming a D_8-tritangent. Moreover they are fixed under symmetric monodromy.

Proof. Since each symmetric cubic surface is a linear combination of elementary homogeneous symmetric polynomials, it suffices to verify each of these vanishes on the lines in the D_8-tritangent, which is a routine computation.

Since symmetric monodromy is S_4-equivariant, the tritangent plane spanned by the lines $\ell_{25}, \ell_{26}, \ell_{27}$ must be stabilized. Moreover, there is no fourth distinct line incident to any symmetric cubic surface which lies in the D_8-tritangent, as this would violate Bézout's theorem. Any nontrivial deformation of $\ell_{25}, \ell_{26}, \ell_{27}$ arising from monodromy would yield such a line, and so the lines $\ell_{25}, \ell_{26}, \ell_{27}$ must be fixed by monodromy within the symmetric locus. □

Observe what this means — given any loop in the symmetric locus, viewed as an element of $W(E_6) \leq S_{27}$, it fixes each of the points 25, 26, and 27. Since $W(E_6)$ acts transitively on ordered tritangents, we can ask what the pointwise stabilizer of a tritangent is in $W(E_6)$, and this will contain our monodromy group.

Proposition 5.9. The symmetric monodromy group is contained in the pointwise stabilizer of a tritangent in $W(E_6)$. This is a group of order 192.

5.2. **Cycle decompositions in $W(E_6)$.** By combining our constraints for the symmetric monodromy group arising from uniformization (Proposition 5.3) and from equivariant enumerative geometry (Proposition 5.9), we obtain the following reduction.

Proposition 5.10. The monodromy group is contained in the group of order 16:

$$\bigcap_{i=25}^{27} \mathrm{Stab}_{W(E_6)}(\ell_i) \cap N_{W(E_6)}(S_4) \cong K_4 \times K_4.$$

We give names to these generators. The former is $K_4 = \langle \sigma_1, \sigma_2 \rangle$, and it is a subgroup of S_4. The latter is $K_4 = \langle \tau_1, \tau_2 \rangle$ and it is not contained in S_4. As explicit elements in $W(E_6) \leq S_{27}$ they are listed in Data A.4

Proposition 5.11. Let $g \in N_{W(E_6)}(S_4) \setminus S_4$, let ℓ be a line on a symmetric cubic surface X, and let $\sigma \in S_4$. Then

$$g(\sigma\ell) = \sigma(g\ell).$$

Proof. By Proposition 2.9, any such element g centralizes S_4 within $W(E_6)$, so σg and $g\sigma$ act identically on any line $\ell \subset X$. □

Corollary 5.12. Let g be any element in the monodromy group, and let ℓ be a line on a symmetric cubic surface X. Then

(1) If g fixes ℓ, then g fixes $\sigma\ell$ for all $\sigma \in S_4$.
(2) If $g\ell \neq \ell$ then $g(\sigma\ell) \neq g(\sigma\ell)$ for all $\sigma \in S_4$.

In particular the monodromy group acts on entire orbits simultaneously.

Proof. If $g \in S_4$ this is clear. If $g \notin S_4$ this follows from Proposition 5.11. □

Proposition 5.13. The symmetric monodromy action does not change the isotropy group of any line. Phrased differently, it acts on each S_4-orbit of lines independently.

Proof. For D_8 this is clear since symmetric monodromy stabilizes each line, so they remain in a D_8 orbit. To see that the S_4/C_2^o and S_4/C_2^e orbits cannot be interchanged by the monodromy action, it suffices to observe that C_2^o and C_2^e are not conjugate in $W(E_6)$. □

Lemma 5.14. If the monodromy group contains two distinct non-trivial elements (i.e. if it is not trivial or cyclic of order two), then it is the Klein 4-group

$$K_4 \cong \langle \sigma_1\tau_2, \tau_1 \rangle .$$

Proof. We can look through the 16 elements of $\langle \sigma_1, \sigma_2, \tau_1, \tau_2 \rangle$ and ask whether each element satisfies the necessary constraints to lie in the symmetric monodromy group.

Since each element in the larger group is an involution, it is a product of disjoint transpositions, the number of which is well-defined in this monodromy problem (since any other choice of basepoint for monodromy conjugates the permutation representation of the monodromy class by an element in $W(E_6)$, but the transposition length of an involution is clearly independent under such a conjugation). We can eliminate the following elements directly for having transposition length 10:

$$\sigma_2\tau_1, \tau_2,\ \sigma_1\sigma_2,\ \sigma_1,\ \sigma_1\sigma_2\tau_1\tau_2,\ \tau_1\tau_2,\ \sigma_2.$$

The remaining non-identity elements have transposition length 6 or 12, so we can ask whether they fix or permute each element of the two orbits. We can eliminate the following three elements, all of transposition length six, since they violate Corollary 5.12:

$$\tau_2,\ \sigma_2\tau_2,\ \sigma_1\sigma_2\tau_2.$$

Therefore our desired monodromy group M is contained in what is left over, meaning we have a *subset* inclusion

$$M \subseteq \{\mathrm{id},\ \tau_1,\ \sigma_2\tau_1,\ \sigma_1\tau_2,\ \sigma_1\tau_1,\ \sigma_1\tau_1\tau_2,\ \sigma_1\sigma_2\tau_1\} .$$

If the monodromy group is non-trivial, then it might be cyclic of order two generated by any one non-identity element in this set. The other case is that it contains two non-identity elements, in which case it must also include their product. In this case, we can exclude those elements in the set whose product with any non-identity element lands outside the set. These elements are $\sigma_2\tau_1$, $\sigma_1\tau_1$, and $\sigma_1\sigma_2\tau_1$. We are left with the Klein 4-group

$$K_4 = \{\text{id},\ \tau_1,\ \sigma_1\tau_2,\ \sigma_1\tau_1\tau_2\}.$$

These elements are explicitly given as permutations in Data A.5. □

5.3. **Certified tracking.** Via Lemma 5.14, our computation of the monodromy group reduces to demonstrating the existence of two loops in the moduli space of smooth symmetric cubic surfaces which induce different permutations on the 27 lines. In conversations with T. Brysiewicz, working with his Pandora software [Bry24], we were able to generate strong evidence that this is true, and determine candidate loops.

Algorithms used in this and related software fall under the umbrella of *homotopy continuation*. This is a key technique in numerical algebraic geometry which deforms a system of polynomial equations along a one-parameter path. One of the primary applications of this technology is conducting explicit monodromy computations.

While homotopy continuation software can generate strong evidence towards a computation, more refined algorithms are needed to turn these computations into proof. At each stage of tracking solutions along a one-parameter path, a guarantee is needed that paths don't collide, and therefore that the computed permutation is indeed correct. These more sophisticated (and time-costly) methods are called *certified tracking algorithms*. Recent work of T. Duff and K. Lee provides algorithms which, among other things, are applicable for certifying computations in monodromy, bridging the gap between computation and proof [DL24, Theorem 1]. In conversations with Lee, their software is able to mathematically certify the following result.

Lemma 5.15 (Numerical certification)**.** There are two loops in the symmetric locus which induce distinct non-identity permutations on the set of 27 lines.

We can now prove Theorem 1.2:

Proof of Theorem 1.2. Lemma 5.14 tells us that the monodromy group M is contained in a specific $K_4 < W(E_6)$. By Lemma 5.15, there are two loops in the symmetric locus which generate distinct non-trivial elements in the monodromy group. Thus these two loops are a generating set for K_4. □

5.4. **The incidence variety of 27 lines over the symmetric locus.** Now that we have determined the monodromy group of the cover $\widetilde{\mathcal{S}} \to \mathcal{S}$ is K_4, we can actually say more about the topology of the space of symmetric cubic surfaces with a line:

Corollary 5.16. The incidence variety of 27 lines restricted to the symmetric locus has 12 connected components. Explicitly as a K_4-set, the fiber over any symmetric cubic surface is of the form

$$6\,[K_4/C_2] + 3\,[K_4/e] + 3\,[K_4/K_4]\,.$$

Proof. Having restricted the symmetric monodromy group and concluding that it is $K_4 < W(E_6)$, we can then see explicitly how K_4 acts on and stablizes the 27 lines on the Fermat cubic surface. This splits them into the 12 families claimed (see Data A.4 for the relevant generators and how they act on the 27 lines). □

To conclude, we have an equivariant analog of Proposition 2.8:

Theorem 5.17. The space $\widetilde{\mathcal{S}}$ is a naturally a (disconnected) complex manifold. Moreover, the equivariant monodromy representation

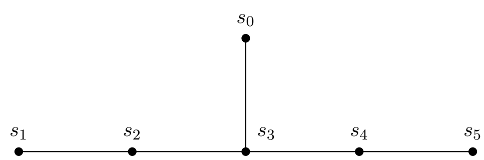
$$\pi_1(\mathcal{S}, X) \to \mathrm{Aut}_{S_4}(H^2(X,\mathbb{Z}), \eta_X) \cong S_4 \times K_4$$

is *not* surjective and has image isomorphic the Klein 4-group K_4.

6. Symmetry and monodromy via representation theory

In the previous section, we determined the monodromy group K_4 within the Weyl group $W(E_6)$ in terms of how it acts on the lines of the Fermat cubic surface; the S_4-orbits of the 27 lines are given in Data A.1. The goal of this section is to understand these copies of S_4 and K_4 in $W(E_6)$ from more traditional representation theoretic viewpoint, via reflection group theory and the projective orthogonal perspective. Informally, we will show that the symmetry group and monodromy group are not visible from purely Coxeter-theoretic considerations.

6.1. **The Weyl group as a reflection group.** To present the Weyl group $W(E_6)$ as a reflection group, we first label the nodes of E_6 Dynkin diagram with the generating reflections $s_0, \ldots, s_5$:



This gives rise to a presentation of the Weyl group of E_6 as a Coxeter group:

$$W(E_6) = \langle s_0, \ldots, s_5 | (s_i s_j)^{m_{ij}} = 1 \rangle, \qquad m_{ij} = \begin{cases} 1 & i = j \\ 3 & s_i,\ s_j \text{ share an edge} \\ 2 & \text{otherwise} \end{cases}$$

Proposition 6.1. Any choice of six skew lines gives rise to a presentation of $W(E_6)$ in the form above.

Proof. The choice of six skew lines determine a marking of the homology of a cubic surface S, where each line corresponds to the homology classes of orthogonal (-1)-exceptional curves $e_1, \ldots, e_6$ on S. These in turn give us a basis of long roots for the E_6 lattice $v_0 = h - e_1 - e_2 - e_3, v_j = e_j - e_{j+1}$ for $j = 1, \ldots, 5$. The intersection form Q on the homology $H_2(S, \mathbb{Z})$ satisfies

$$Q(h,h) = 1$$
$$Q(e_i, e_j) = -\delta_{ij}$$
$$Q(h, e_i) = 0.$$

From this it is clear that $Q(v_i, v_i) = -2$ for any $0 \leq i \leq 5$. Then the reflections s_i that generate the Weyl group $W(E_6)$ are realized homologically by

$$s_i(x) = x - \frac{2Q(x, v_i)}{Q(v_i, v_i)} v_i = x + Q(x, v_i) v_i;$$

this is the *geometric representation* of $W(E_6)$. □

Example 6.2. If we pick the lines $[\ell_1, \ell_3, \ell_{10}, \ell_{11}, \ell_{16}, \ell_{22}]$, a direct computation gives the six generators of $W(E_6)$ as the following permutations in S_{27}:

s_0	`(1,8)(3,6)(9,26)(10,25)(13,21)(20,23)`
s_1	`(1,3)(2,4)(5,7)(6,8)(14,15)(18,19)`
s_2	`(2,12)(3,10)(5,27)(6,25)(14,17)(19,24)`
s_3	`(5,8)(6,7)(9,12)(10,11)(17,20)(21,24)`
s_4	`(5,14)(7,15)(9,13)(11,16)(17,27)(21,26)`
s_5	`(13,23)(14,19)(15,18)(16,22)(17,24)(20,21).`

It is a classical computation that there are exactly 72 ways to pick six pairwise skew lines on a cubic surface.

6.2. **Double sixes from the Weyl group.** Given six ordered pairwise skew lines, we obtain an associated subgroup $W(A_5) \leq W(E_6)$ by suppressing the node s_0, and all of these subgroups are conjugate. We note though, that we can permute the ordering of our six lines – a natural question to ask is whether such a permutation extends to element of the Weyl group, and if such an extension exists, whether it is unique. The answer to both these questions is yes.

Proposition 6.3. Given six ordered skew lines, any automorphism σ of them extends uniquely to an adjacency-preserving automorphism of all 27 lines, i.e. an element of $W(E_6)$.

Proof. Any permutation of the lines permutes the homology classes $e_1, \dots, e_6$ accordingly, and in particular will fix the canonical class $K_S = 3h - e_1 - \cdots - e_6$. Therefore by definition it extends to an element of $W(E_6)$. Since its action on the e_i's defines its action on h and therefore on a basis of the homology, such an extension is unique. □

Moreover, we understand this subgroup of $W(E_6)$.

Proposition 6.4. Fixing six ordered skew lines, the subgroup of $W(E_6)$ obtained by permuting them is exactly equal to the Weyl group $W(A_5)$ obtained from the presentation coming from the choice of lines.

Proof. It suffices to show that each of the generators $s_1, \dots, s_5$ is contained in this symmetric group. This is immediate, since s_i permutes e_i and e_{i+1} and fixes the other e_j. □

There is a unique conjugacy class of subgroup $W(A_5) \leq W(E_6)$, and $W(E_6)/W(A_5)$ is a transitive set of order 36. There are, however, 72 unordered choices of six skew lines. This gives us a surjection

$$\{\text{six skew lines}\} \to W(E_6)/W(A_5),$$

which is 2-to-1. In particular, six skew lines come in pairs, which give rise to the same copy of $W(A_5)$ in $W(E_6)$. These pairs of six skew lines are what are known as *double sixes.*

In particular a computation shows that, as a $W(A_5)$-set, the set of lines $\{1, \dots, 27\}$ decompose into two transitive $W(A_5)$-sets of order six, and a single transitive set of order 15. These are the double six, and the remaining lines, respectively.

Remark 6.5. While $W(A_5)$ is isomorphic to S_6 as we have seen, it is abuse of terminology to equate them. There are *two* non-conjugate subgroups of $W(E_6)$ which are isomorphic to S_6, the first being our $W(A_5)$ group, and the latter just being another subgroup of $W(E_6)$ which we denote by S_6. The latter group can be distinguished via its action on 27 lines — it acts transitively on 12 lines and transitively on the other 15.

6.3. **Our groups are not reflection groups.** We can now argue that both the S_4 acting on symmetric cubic surfaces and the symmetric monodromy group are *not reflection subgroups of* $W(E_6)$. This is perhaps obvious to those familiar with e.g. [Man06], but we can give an elementary argument now with the machinery we have built.

Proposition 6.6. The subgroup $S_4 \leq W(E_6)$ is not a reflection subgroup — that is, it is not isomorphic to $W(A_3)$ for a presentation of $W(E_6)$ arising from any choice of six skew lines.

Proof. We prove something stronger, namely that S_4 is not subconjugate to $W(A_5)$. Indeed suppose towards a contradiction that it was. As we have seen by [Bra24], the action of S_4 on the 27 lines decomposes into three S_4-sets, of order 12, 12, and 3. If $S_4 \leq W(A_5)$, then this action would be restricted from the action of $W(A_5)$ on the set of 27 lines. However the partition of $\{1, \ldots, 27\}$ into orbits will only ever *refine* under a restricted group action. In particular since $W(A_5)$ has two orbits of size six it cannot restrict to the prescribed S_4-action. □

Remark 6.7. The action of the *other* S_6 from Remark 6.5 does not have this same restriction, and a computation shows that S_4 is indeed subconjugate to S_6 in $W(E_6)$.

Remark 6.8.

(1) Another interesting note is that while S_4 is not subconjugate to $W(A_5)$, we have that $W(A_5)$ is nested in a maximal subgroup isomorphic to $W(A_5) \times C_2 \leq W(E_6)$. It *is* true that S_4 is subconjugate to this maximal subgroup, and moreover the centralizer of S_4 in $W(A_5) \times C_2$ is identical to the symmetric monodromy group!

(2) There is actually a *unique* copy of $W(A_5)$ in $W(E_6)$ for which S_4 is a subgroup of its maximal supergroup $W(A_5) \times C_2$. This unique copy corresponds to a *preferred double six for symmetric cubic surfaces*. A direct computation shows that this is the unique double six where six skew lines lie in the same S_4-orbit.

Proposition 6.9. The symmetric monodromy group $K_4 \leq W(E_6)$ is not a reflection subgroup.

Proof. Suppose for the sake of contradiction that K_4 was a reflection subgroup; it would then take on the form of $W(A_1) \times W(A_1)$. Since each of the generators s_i act on the 27 lines as a product of six disjoint transpositions, there are two nontrivial elements of $W(A_1) \times W(A_1)$ that are the product of six disjoint transpositions. However, Data A.5 tells us that the symmetric monodromy group only has one element that is the product of six disjoint transpositions, a contradiction. □

6.4. **Symmetric monodromy in the projective orthogonal groups.** Since we know how the symmetric monodromy group $K_4 < W(E_6)$ acts on the 27 lines of the Fermat cubic surface S, we can explicitly connect this K_4 back to the projective orthogonal group by a lengthy homological calculation. We sketch this correspondence now.

Recall that the set of six skew lines $\{\ell_1, \ell_3, \ell_{10}, \ell_{11}, \ell_{16}, \ell_{22}\}$ determine a marking of the homology of S, where each line corresponds to the homology classes of orthogonal (-1)-exceptional curves $e_1, e_2, e_3, e_4, e_5, e_6$ on S. Using Data A.5, we can calculate how $K_4 = \langle \tau_1, \sigma_1\tau_2 \rangle$ acts on the exceptional (-1)-curves, which in turns explicitly determines how K_4 acts on the E_6 lattice. Then by passing to the root lattice quotient used in the proof of the exceptional isomorphism outlined in Proposition 3.3, this K_4 projects to the symmetric monodromy group K_4 inside of $\mathrm{PO}(4, 1, \mathbb{F}_3)$.

It would be interesting to understand how the symmetric monodromy group arises purely by an analyzing its action on the associated symmetric cyclic cubic 3-folds. This leads us to the following problem:

Problem 6.10. Determine the symmetric monodromy group K_4 as a subgroup $\mathrm{PO}(4, 1, \mathbb{F}_3)$ directly, that is, without reference to the action on the lines or the exceptional isomorphism with $W(E_6)$.

As Beauville remarks [Bea09, pg. 19], what makes this difficult is that it is unknown how to produce a marking of a cubic surface from a framing of the corresponding cyclic cubic 3-fold. A resolution to this problem would shed further light on symmetric monodromy can be witnessed by Hodge theory, and therefore be applied to similar equivariant enrichments of classical enumerative problems.

Appendix A. Data tables

We record some of the line geometry data associated to the Fermat cubic surface.

A.1. **All about the Fermat.**

Data A.1. The 27 lines ℓ_i on the Fermat can be labeled and grouped according to their S_4-orbits as follows:

i	ℓ_i
1	$[w, -w, z, \zeta \cdot z]$
2	$[w, -w, z, \zeta^5 \cdot z]$
3	$[w, \zeta \cdot w, z, -z]$
4	$[w, \zeta^5 \cdot w, z, -z]$
5	$[w, z, \zeta \cdot w, -z]$
6	$[w, z, \zeta^5 \cdot w, -z]$
7	$[w, z, -w, \zeta \cdot z]$
8	$[w, z, -w, \zeta^5 \cdot z]$
9	$[w, z, -z, \zeta \cdot w]$
10	$[w, z, -z, \zeta^5 \cdot w]$
11	$[w, z, \zeta \cdot z, -w]$
12	$[w, z, \zeta^5 \cdot z, -w]$

i	ℓ_i
13	$[w, \zeta \cdot w, z, \zeta \cdot z]$
14	$[w, \zeta \cdot w, z, \zeta^5 \cdot z]$
15	$[w, \zeta^5 \cdot w, z, \zeta \cdot z]$
16	$[w, \zeta^5 \cdot w, z, \zeta^5 \cdot z]$
17	$[w, z, \zeta \cdot w, \zeta \cdot z]$
18	$[w, z, \zeta \cdot w, \zeta^5 \cdot z]$
19	$[w, z, \zeta^5 \cdot w, \zeta \cdot z]$
20	$[w, z, \zeta^5 \cdot w, \zeta^5 \cdot z]$
21	$[w, z, \zeta \cdot z, \zeta \cdot w]$
22	$[w, z, \zeta^5 \cdot z, \zeta \cdot w]$
23	$[w, z, \zeta \cdot z, \zeta^5 \cdot w]$
24	$[w, z, \zeta^5 \cdot z, \zeta^5 \cdot w]$

i	ℓ_i
25	$[w, -w, z, -z]$
26	$[w, z, -w, -z]$
27	$[w, z, -z, -w]$

Data A.2. Given the labeling of the lines on the Fermat as in Data A.1, the Galois group $W(E_6)$ is given by

```
G := SymmetricGroup(27);
W:= Subgroup(G,[
(13,23)(14,19)(15,18)(16,22)(17,24)(20,21),
(5,14)(7,15)(9,13)(11,16)(17,27)(21,26),
(2,6)(4,8)(5,19)(7,18)(9,23)(11,20)(12,25)(16,21)(22,26)(24,27),
(5,8)(6,7)(9,12)(10,11)(17,20)(21,24),
(3,4)(5,10)(6,9)(7,12)(8,11)(13,15)(14,16)(17,24)(18,23)(19,22)(20,21)(26,27),
(1,2)(5,9)(6,10)(7,11)(8,12)(13,14)(15,16)(17,21)(18,22)(19,23)(20,24)(26,27)
]);
```

Data A.3. The S_4-action on the 27 lines of the Fermat cubic surface, given by permuting coordinates on $\mathbb{CP}^3$, are generated by the following transposition and 4-cycle:

elt	permutation
transp.	(3,4)(5,11)(6,12)(7,9)(8,10)(13,15)(14,16)(17,21)(18,23)(19,22)(20,24)(26,27)
4-cycle	(1,11,3,10)(2,12,4,9)(5,8,6,7)(13,23)(14,24, 15,21)(16,22)(17,18,20,19)(25,27)

Data A.4. The generators $\sigma_1, \sigma_2, \tau_1, \tau_2 \in W(E_6)$ from Proposition 5.10 are given by the following permutations:

elt	permutation
σ_1	(1,3)(2,4)(5,6)(7,8)(9,12)(10,11)(14,15)(17,20)(18,19)(21,24)
σ_2	(1,4)(2,3)(5,8)(6,7)(9,10)(11,12)(13,16)(17,20)(21,24)(22,23)
τ_1	(13,23)(14,19)(15,18)(16,22)(17,24)(20,21)
τ_2	(1,4)(2,3)(9,11)(10,12)(13,16)(22,23)

Data A.5. The (non-identity) elements in the Klein 4-group corresponding to symmetric monodromy are given by

elt	permutation
τ_1	(13,23)(14,19)(15,18)(16,22)(17,24)(20,21)
$\sigma_1\tau_2$	(1,3)(2,4)(5,6)(7,8)(9,12)(10,11)(13,23)(14,18)(15,19)(16,22)(17,21)(20,24)
$\sigma_1\tau_1\tau_2$	(1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,22)(14,18)(15,19)(16,23)(17,21)(20,24)

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