HOPF ALGEBROIDS

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Abstract. Notes from an expository talk given in the UPenn chromatic homotopy theory seminar, spring 2021.

1. Basic definitions

Let $\mathscr{C}$ be a category with finite products. Then a *group object* in $\mathscr{C}$ is an element $G \in \mathscr{C}$ together with maps

$$\begin{aligned} m : G \times G &\to G \\ e : 1 &\to G \\ i : G &\to G, \end{aligned}$$

multiplication, unity, and inversion, respectively, which satisfy the expected axioms.

Proposition 1.1. Let G be a group object in $\mathscr{C}$, and assume that $\mathscr{C}$ is locally small. Then for any $X \in \mathscr{C}$, we have that

$$\mathrm{Hom}_{\mathscr{C}}(X, G)$$

is a group. Here the group operation is given by

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, G) \times \mathrm{Hom}_{\mathscr{C}}(X, G) &\to \mathrm{Hom}_{\mathscr{C}}(X, G) \\ (f, h) &\mapsto m \circ (f \times h). \end{aligned}$$

This leads us to a different definition. A *group object structure* on an element $G \in \mathscr{C}$ is an extension

$$\begin{array}{ccc} & & \mathtt{Grp} \\ & \nearrow & \downarrow U \\ \mathscr{C}^{\mathrm{op}} & \xrightarrow[\mathrm{Hom}(-,G)]{} & \mathtt{Set}. \end{array}$$

That is, it is an element, together with some additional data, that represents a functor $\mathscr{C}^{\mathrm{op}} \to \mathtt{Grp}$.

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Suppose we want to categorify this a bit – so instead of the category `Grp` of groups, we want to work with the category `Grpd` of *groupoids*. That is, suppose $\mathscr{C}$ is a locally small category, and `Grpd` is the category of (small) groupoids. Consider a functor

$$F : \mathscr{C}^{\text{op}} \to \texttt{Grpd}.$$

What data do we need to represent it? We can't claim that it is representable by a single object anymore — given an object $X \in \mathscr{C}$, for any other object Y we will have that $\text{Hom}_{\mathscr{C}}(Y, X)$ is a set; that is, a discrete category. There is no nice way to view this set as anything but a discrete groupoid in its own right.

The idea is to take *two objects* in $\mathscr{C}$, let's call them A and Γ, so that for any element $Y \in \mathscr{C}$, we have that $F(Y)$ is a groupoid, with objects and morphisms given by

$$\begin{aligned} \text{ob}F(Y) &= \text{Hom}_{\mathscr{C}}(Y, A) \\ \text{mor}F(Y) &= \text{Hom}_{\mathscr{C}}(Y, \Gamma). \end{aligned}$$

If you were handed two loose sets and told that one was the objects of a category and the other was the morphisms, you might say "thanks for nothing." We need a way to tell which morphisms were traveling between which objects. That is, we should have source and target maps

$$s, t : \text{mor}F(Y) \to \text{ob}F(Y),$$

that is,

$$s, t : \text{Hom}_{\mathscr{C}}(Y, \Gamma) \to \text{Hom}_{\mathscr{C}}(Y, A).$$

Since we don't really want these to depend on Y in any way, it might be natural to ask that these come from post-composition with morphisms $\Gamma \to A$. To that end, we define two maps, which by abuse of notation we also call *source* and *target*

$$s, t : \Gamma \to A.$$

We also need identities – that is, for every object in $F(Y)$ there is a unique way to assign it an identity morphsim. This can be thought of as a map

$$\text{Hom}_{\mathscr{C}}(Y, A) = \text{ob}F(Y) \to \text{mor}F(Y) = \text{Hom}_{\mathscr{C}}(Y, \Gamma).$$

Again we want this to be independent of Y, so we could ask for it to come from post-composition with a morphism

$$i : A \to \Gamma.$$

Already we are forced to ask for some coherence between these things. The source of the identity morphism is the object you started with, so the following diagram must commute

$$\begin{array}{ccc} \text{ob}F(Y) & \xrightarrow{\text{identity}} & \text{mor}F(Y) \\ & \searrow^{=} & \downarrow \text{source} \\ & & \text{ob}F(Y), \end{array}$$

which in our language corresponds to the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & \Gamma \\ & \searrow^{\mathrm{id}_A} & \downarrow s \\ & & A. \end{array}$$

Similarly, we want that $t \circ i = \mathrm{id}_A$.

We also want composition, which will (almost) be of the following form:

$$\mathrm{Hom}(Y, \Gamma) \times \mathrm{Hom}(Y, \Gamma) \to \mathrm{Hom}(Y, \Gamma).$$

There is an issue here, which is that we don't want any two morphisms to have a composite, only *composable* morphisms. That is, the types of morphisms we want are pairs (f, g) where the source of f is the target of g. That is, our composition should be

$$\mathrm{Hom}(Y, \Gamma) \times_{\mathrm{Hom}(Y,A)} \mathrm{Hom}(Y, \Gamma) \to \mathrm{Hom}(Y, \Gamma).$$

In order to make this natural, we *assume that $\mathscr{C}$ has finite pullbacks.* Then composition is of the form

$$\mathrm{Hom}(Y, \Gamma \times_A \Gamma) \to \mathrm{Hom}(Y, \Gamma),$$

where $\Gamma \times_A \Gamma$ is the pullback

$$\begin{array}{ccc} \Gamma \times_A \Gamma & \longrightarrow & \Gamma \\ \downarrow & \lrcorner & \downarrow s \\ \Gamma & \xrightarrow[t]{} & \Gamma. \end{array}$$

For naturality, we ask that composition comes from a map

$$m : \Gamma \times_A \Gamma \to \Gamma.$$

We ask that composition is associative:

$$\begin{array}{ccc} \Gamma \times_A \Gamma \times_A \Gamma & \xrightarrow{m \times \mathrm{id}_\Gamma} & \Gamma \times_A \Gamma \\ {\scriptstyle \mathrm{id} \times m}\downarrow & & \downarrow m \\ \Gamma \times_A \Gamma & \xrightarrow[m]{} & \Gamma. \end{array}$$

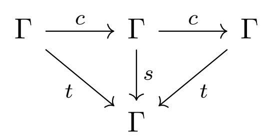
And we ask that composing with the identity on the left or right doesn't do anything

$$\begin{array}{ccc} A \times_A \Gamma & \xrightarrow{i \times \mathrm{id}_\Gamma} & \Gamma \times_A \Gamma \\ & \searrow^{\cong} & \downarrow m \\ & & \Gamma \end{array} \qquad \begin{array}{ccc} \Gamma \times_A A & \xrightarrow{\mathrm{id}_\Gamma \times i} & \Gamma \times_A \Gamma \\ & \searrow^{\cong} & \downarrow m \\ & & \Gamma. \end{array}$$

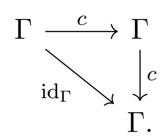
Stopping here, we've successfully represented a functor to (small) categories! That is, the data above tells you how to represent a functor $F : \mathscr{C} \to \mathtt{Cat}$. In order to deal with groupoids, we have to confront the existence of inverses.

To any morphism, you can uniquely associate its inverse. This should come from a morphism $c : \Gamma \to \Gamma$. This forces us into three more coherence conditions:

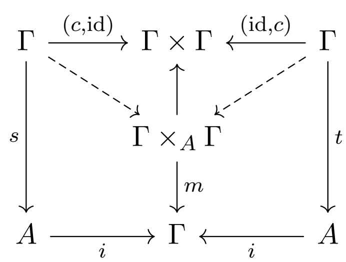
(1) *Inverting a morphism swaps source and target*:



(2) *Inverting twice does nothing*



(3) *Composing a morphism with its inverse gives the identity* – this is strange because the pair of maps $i, c : \Gamma \to \Gamma$ give a map $\Gamma \to \Gamma \times \Gamma$, not to the fiber product. So we ask for dashed maps making the following diagram commute



This data tells you how to represent a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathtt{Grpd}$.

How do we know that we're done? Why couldn't there be other coherence conditions?

Exercise 1.2. Check that if $\mathscr{C} = \mathtt{Set}$, then the data above specifies a small groupoid.

2. Digression on stacks

Suppose $\mathscr{C}$ is a category with a Grothendieck topology, and we have a functor

$$F : \mathscr{C}^{\mathrm{op}} \to \mathtt{Set}.$$

Then F is a *sheaf* if for any cover $\{U_i \to U\}$, we have an equalizer sequence

$$F(U) \to \prod F(U_i) \rightrightarrows \prod F(U_i \times_U U_j).$$

This tells us that we can glue elements in $F(U_i)$ that agree on overlaps to get an element in $F(U)$, and that elements in $F(U)$ are determined by what they look like locally.

Set-valued sheaves automatically satisfy higher cocycle conditions, so it is equivalent to say that

$$\begin{aligned} F(U) &= \lim\left(\prod F(U_i) \rightrightarrows \prod F(U_i \times_U U_j)\right) \\ &= \lim\left(\prod F(U_i) \rightrightarrows \prod F(U_i \cap U_j) \rightrightarrows \prod (F(U_i \times_U U_j \times_U U_k) \to \cdots\right) \end{aligned}$$

There is a sense [Dug] in which the category $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathtt{Set})$ can be given a model structure where sheaves are the fibrant objects. Then every presheaf can be turned into a sheaf by fibrant replacement. This is called *sheafification.*

Now suppose that F is a functor

$$F : \mathscr{C}^{\mathrm{op}} \to \mathtt{Grpd}.$$

We can ask a similar question — does local data glue to global data in a coherent way? If so, we will call it a stack. It turns out we can impose the same condition as above, but here wecan't quite glue things directly. Instead we have to glue them in a looser way. This is imposed by homotopy limits.

Definition 2.1. [Hol08, p. 1.3] A presheaf of groupoids $F : \mathscr{C}^{\mathrm{op}} \to \mathtt{Grpd}$ is a *stack* if for every cover $\{U_i \to U\}_i$, we have an equivalence of groupoids

$$F(U) \xrightarrow{\sim} \mathrm{holim}\left(\prod F(U_i) \rightrightarrows \prod F(U_i \times U_j) \to \cdots\right).$$

The model structure here is on $\mathtt{Grpd}$, and it is:

- weak equivalences = equivalences of categories
- fibrations have RLP with respect to $\Delta^0 \to \Delta^1$
- cofibrations are objectwise injections.

This is left proper, simplicial, cofibrantly generated [Hol08, p. 2.1].

Again, there is a model structure on $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathtt{Grpd})$ in which the stacks are the fibrant objects, so every presheaf of groupoids can be turned into a stack via fibrant replacement, called *stackification* [Hol08, p. 1.2].

Example 2.2. Every sheaf of sets gives a sheaf of discrete groupoids, which is a stack.

Suppose we have a representable functor $F : \mathscr{C}^{\mathrm{op}} \to \mathtt{Grpd}$, represented by a groupoid object (A, Γ). Then it can be stackified!

$$\{\text{groupoid objects on } \mathscr{C}\} \rightsquigarrow \{\mathtt{Grpd}\text{-valued presheaves on } \mathscr{C}\} \rightsquigarrow \{\text{stacks on } \mathscr{C}\}.$$

Example 2.3. Let G be a topological group. Consider the functor

$$\begin{aligned} \mathtt{Top}^{\mathrm{op}} &\to \mathtt{Grpd} \\ X &\mapsto \mathrm{Prin}_G(X), \end{aligned}$$

sending X to the groupoid of principal G-bundles. This is represented by the pair $(*, G)$. The associated stack is $\mathcal{B}G$, the classifying *stack* of the group G.

Example 2.4. Let $F : \mathtt{Top}^{\mathrm{op}} \to \mathtt{Grpd}$ be the functor sending a connected space Y to the groupoid $F(Y)$ whose objects are maps $Y \to X$, and whose isomorphisms are commutative diagrams of the form

$$\begin{array}{ccc} Y & \xrightarrow{f_1} & X \\ & \searrow^{f_2} & \downarrow{\scriptstyle g\cdot} \\ & & X. \end{array}$$

In particular, we see that $F(*)$ is just the translation groupoid $\mathcal{E}X$. We claim this functor is representable. That is, there are functors

$$\mathrm{Hom}_{\mathtt{Top}}(-, X) : \mathtt{Top}^{\mathrm{op}} \to \mathtt{Set}$$
$$\mathrm{Hom}_{\mathtt{Top}}(-, G \times X) : \mathtt{Top}^{\mathrm{op}} \to \mathtt{Set},$$

with some coherence maps between X and $G \times X$, representing F. That is, the pair $(X, G \times X)$ is a *groupoid object* in spaces. We can stackify this, and we obtain $\mathcal{M}_{(X, G\times X)}$. This is the *orbifold* associated to G acting on X.

3. Hopf algebroids

Definition 3.1. Let K be a commutative ring. A *Hopf algebroid* is a groupoid object in the category of affine K-schemes.

Under the anti-equivalence of categories

$$\mathit{Aff}_K \xrightarrow{\sim} \mathtt{CAlg}_K^{\mathrm{op}},$$

we may consider a Hopf algebroid as a *cogroupoid object* in the category of commutative K-algebras.

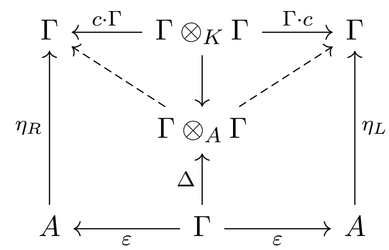
Definition 3.2. [Rav86, A.1] A *Hopf algebroid* over K is a pair (A, Γ) of elements in $\mathtt{CAlg}_K$, together with maps

map	**definition**	**categorical interpretation**
$\eta_L : A \to \Gamma$	exhibiting $\Gamma \in {}_A\mathtt{Mod}$	target
$\eta_R : A \to \Gamma$	exhibiting $\Gamma \in \mathtt{Mod}_A$	source
$\Delta : \Gamma \to \Gamma \times_A \Gamma$	coproduct	composition
$\varepsilon : \Gamma \to A$	counit	identity
$c : \Gamma \to \Gamma$	conjugation	inverse,

satisfying the following axioms:

(1) $\varepsilon\eta_L = \varepsilon\varepsilon_R = 1_A$ (source and target of an identity)
(2) $(\Gamma \otimes \varepsilon)\,\Delta = (\varepsilon \otimes \Gamma)\,\Delta = 1_\Gamma$ (composition with an identity)
(3) $(\Gamma \otimes \Delta)\,\Delta = (\Delta \otimes \Gamma)\,\Delta$ (composition is associative)
(4) $c\eta_R = \eta_L$ and $\eta_L = c\eta_R$ (inverting a morphism interchanges source and target)
(5) $cc = 1_\Gamma$ (inverse of inverse)

(6) There are maps making the diagram commute:



Example 3.3. There is a functor

$$E : \texttt{Ring} \to \texttt{Grpd},$$

sending a ring R to the groupoid $E(R)$, whose objects are explicit formulas for elliptic curves with coefficients in R, and whose morphisms are reparametrizations. This functor is representable, that is, it is given by a Hopf algebroid (A, Γ). The associated stack $\mathcal{M}_{(A,\Gamma)}$ is the moduli stack of elliptic curves. For details, see [HM14; Mat].

Example 3.4. Recall that for any spectrum E, its *homology* is defined by

$$E_*(X) := \pi_*(E \wedge X) = [\mathbb{S}, E \wedge X].$$

Thus the homology of itself is $E_*E = [\mathbb{S}, E \wedge E]$. It also makes sense to denote by $E_* = E_*(*) = \left[\Sigma^\infty S^0, E \wedge S^0\right] = [\mathbb{S}, E] = \pi_*E$.

If E is a commutative ring spectrum, with some unit map $\mathbb{S} \to E$, then there is an induced map

$$E = \mathbb{S} \wedge E \to E \wedge E.$$

Applying π_*, we get a ring homomorphism

$$E_* = \pi_*E \to \pi_*\left(E \wedge E\right) = E_*E.$$

This exhibits $E_*(E)$ as a left π_*E-module. Dually, we can take $E = E \wedge \mathbb{S} \to E \wedge E$ to get a right module structure. So the TL;DR here is that *E_*E is a π_*E-bimodule.*

By smashing with the unit $u : \mathbb{S} \to E$ on the left or right, we obtain two maps $E \to E \wedge E$. Applying π_*, we have

$$\eta_L, \eta_R : E_* \to E_*E.$$

Proposition 3.5. We have that η_L is a flat ring homomorphism if and only if η_R is. Recall that means that

$$E_*E \otimes_{\pi_*E} (-) : \texttt{Mod}_{\pi_*E} \to \texttt{Mod}_{\pi_*E}$$

is exact. In this situation, we say that E is *flat* as a ring spectrum. This is often satisfied.

Proposition 3.6. Let A and B be ring spectra. Then there is a pushout diagram

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & A \\ \downarrow & \ulcorner & \downarrow \\ B & \longrightarrow & A \wedge B. \end{array}$$

Applying π_* won't preserve the pushout, but it will induce a ring homomorphism:

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \pi_* A \arrow[d] \arrow[rdd, bend left] & \\ \pi_* B \arrow[r] \arrow[rrd, bend right] & \pi_* A \otimes \pi_* B \arrow[rd, dashed] & \\ & & \pi_* (A \wedge B). \end{tikzcd}$$

If the unit map $\mathbb{S} \to A$ is an equivalence (e.g. if A is the suspension spectrum of a sphere), then π_* will preserve the pushout, and the natural map

$$\pi_* A \otimes \pi_* B \to \pi_*(A \wedge B)$$

will be a ring isomorphism.

As the tensor product is a coequalizer, we may consider the following commutative diagram, to get an induced map on tensor products, where X is an arbitrary spectrum:

$$\begin{tikzcd} E_* E \otimes E_* \otimes E_* X \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & E_* E \otimes E_* X \arrow[r] \arrow[d] & E_* E \otimes_{E_*} E_* X \arrow[d, dashed] \\ \pi_* (E \wedge E \wedge E \wedge E \wedge X) \arrow[r, shift left] \arrow[r, shift right] & \pi_* (E \wedge E \wedge E \wedge X) \arrow[r] & \pi_* (E \wedge E \wedge X). \end{tikzcd}$$

Proposition 3.7. If E is flat, then the map

$$E_* E \otimes_{E_*} E_* X \to \pi_* (E \wedge E \wedge X)$$

is a ring isomorphism.

Proposition 3.8. For a sufficiently nice E_∞-ring spectrum E, we have that the pair $(E_*, E_* E)$ is a Hopf algebroid.

Proof. We have to define all the data associated to it and then check the desired axioms hold.

(1) By smashing with the unit $u : \mathbb{S} \to E$ on the left or right, we obtain two maps $E \to E \wedge E$. Applying π_*, we have

$$\eta_L, \eta_R : E_* \to E_* E.$$

These are *target* and *source*.

(2) We have a flip map $E \wedge E \to E \wedge E$ which squares to the identity. Applying π_* we get

$$c : E_*E \to E_*E.$$

This is *inverse*.

(3) There is a multiplication map $E \wedge E \xrightarrow{\mu} E$ coming from E being a ring spectrum. Applying π_* we get

$$E_*E \to E_*.$$

This is *counit*.

(4) Finally, we want a coproduct map, which in our terminology will be of the form $E_*E \to (E_*E) \otimes_{\pi_*E} (E_*E)$. Here is where we really require E_*E to be a π_*E-bimodule! From the argument above, we have that

$$E_*E \otimes_{E_*} E_*X \to \pi_*(E \wedge E \wedge X)$$

is an isomorphism. When $X = E$, then there is a natural map

$$E \wedge E = E \wedge S^0 \wedge E \to E \wedge E \wedge E,$$

and by applying π_*, we get a map

$$\Delta : E_*E \to E_*E \otimes_{\pi_*E} E_*E.$$

We can check all the axioms are satisfied (exercise). □

Example 3.9. We have that $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$ is a Hopf algebroid. That is, it corepresents a functor $\mathtt{Ring} \to \mathtt{Grpd}$. We can ask what functor this is. After proving Quillen's theorem, we will have $\mathrm{MU}_* = \pi_*\mathrm{MU} = L$ is the Lazard ring. So $\mathrm{Hom}(\mathrm{MU}_*, -) = \mathrm{Hom}(L, -)$, which represents formal group laws.

It turns out that $\mathrm{Hom}(\mathrm{MU}_*\mathrm{MU}, -)$ will represent reparametrizations of formal group laws (change of bases). So we have that the pair $(\mathrm{MU}_*, \mathrm{MU}_*\mathrm{MU})$ represents the functor

$$\begin{aligned} \mathtt{Ring} &\to \mathtt{Grpd} \\ R &\mapsto \{\text{formal group laws over } R\}. \end{aligned}$$

The *stackification* $\mathcal{M}_{\mathrm{MU}_*,\mathrm{MU}_*\mathrm{MU}}$ will be referred to as the *moduli stack of formal groups*, and denoted $\mathcal{M}_{\mathrm{FG}}$. This has the following property:

$$\mathrm{Hom}_{\mathtt{Stack}}(\mathrm{Spec}(A), \mathcal{M}_{\mathrm{FG}}) \cong \{\text{FGLs over } A\}.$$

This is an isomorphism of groupoids.

4. Modules over a Hopf algebroid

Definition 4.1. Let (A, Γ) be a Hopf algebroid. A *comodule* M over (A, Γ) is a left A-module M with a "coaction" A-module map

$$\eta : M \to \Gamma \otimes_A M,$$

so that

(1) the composite $M \xrightarrow{\eta} \Gamma \otimes_A M \xrightarrow{\varepsilon \otimes 1} M$ is the identity (counitality)
(2) we have that coassociativity holds:

$$\begin{array}{ccc} M & \xrightarrow{\eta} & \Gamma \otimes_A M \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\Delta\otimes M} \\ \Gamma \otimes_A M & \xrightarrow[\mathrm{id}\otimes\eta]{} & \Gamma \otimes_A \Gamma \otimes_A M \end{array}$$

In particular there is always a forgetful functor

$$\texttt{coMod}_{(A,\Gamma)} \to \texttt{Mod}_A.$$

Example 4.2. Let X be any spectrum, and E a flat ring spectrum. Then the map

$$E \wedge X = E \wedge S^0 \wedge X \xrightarrow{1\wedge u \wedge 1} E \wedge E \wedge X$$

induces a map on homotopy

$$E_*X \to E_*E \otimes_{E_*} E_*X.$$

We can check that this satisfies the axioms of a coaction map. Therefore for any spectrum, we get an (E_*, E_*E)-comodule. This is functorial [Rav86, p. 2.2.8]:

$$\begin{aligned} E_* : Sp &\to \texttt{coMod}_{(E_*,E_*E)} \\ X &\mapsto E_*X. \end{aligned}$$

Example 4.3. As a particular case, consider the sphere S^t, viewed as a suspension spectrum. Then its E-homology is

$$E_*(S^t) = \pi_*\left(E \wedge S^t\right) = \pi_*\left(\Sigma^t E\right) = \Sigma^t E_*.$$

So we have that $\Sigma^t E_* \in \texttt{coMod}_{(E_*,E_*E)}$ for each $t \in \mathbb{Z}$.

Example 4.4. Let M be an E_*-module. Then we have that $E_*E \otimes_{\pi_*E} M$ is an (E_*, E_*E)-comodule, called the *extended comodule* where the coaction map is induced by the coproduct on E_*E. This is also functorial, and defines a functor

$$\texttt{Mod}_{\pi_*E} \to \texttt{coMod}_{(E_*,E_*E)}.$$

We call

This turns out to be right adjoint to the forgetful functor

$$\text{forget} : \texttt{coMod}_{(E_*,E_*E)} \leftrightarrows \texttt{Mod}_{E_*} : \text{extend}.$$

If we were in the context of abelian categories, this would give us a lovely way to produce injective objects in $\texttt{coMod}_{(E_*,E_*E)}$. Since right adjoints preserve injective objects, we could just take injective E_*-modules and extend them. It turns out that $\texttt{coMod}_{(E_*,E_*E)}$ will be abelian under pretty weak conditions.

Theorem 4.5. [Rav86, A.1.1.3] If Γ is a flat A-module, then we have that the category of left (A, Γ)-comodules is abelian, with enough injectives.

Proof. Consider a short exact sequence of A-modules

$$0 \to B \to C \to D \to 0.$$

Then since Γ is flat, then $\Gamma \otimes_A -$ is exact, so we have a short exact sequence

$$0 \to \Gamma \otimes_A B \to \Gamma \otimes_A C \to \Gamma \otimes_A D \to 0.$$

If C has a (A, Γ)-comodule structure (by which we mean a map $C \to \Gamma \otimes_A C$ satisfying the properties above) then we can verify that $D \in \mathtt{coMod}_{(A,\Gamma)}$ if and only if $B \in \mathtt{coMod}_{(A,\Gamma)}$. Thus every map of comodules can be seen to have a kernel and cokernel defined in $\mathtt{coMod}_{(A,\Gamma)}$. We can check the other axioms of an abelian category hold. □

Example 4.6. If E is a flat ring spectrum, then we have an abelian category $\mathtt{coMod}_{(E_*,E_*E)}$.

Proposition 4.7. Let (A, Γ) be a Hopf algebroid, and $\mathcal{M}_{(A,\Gamma)}$ the associated stack. There is an equivalence of categories

$$\mathtt{QCoh}\left(\mathcal{M}_{(A,\Gamma)}\right) \simeq \mathtt{coMod}_{(A,\Gamma)}.$$

Let (A, Γ) be any flat Hopf algebroid, and let M be an arbitrary projective comodule. Then there is a functor

$$\mathrm{Hom}_{\mathtt{coMod}_{(A,\Gamma)}}(M, -) : \mathtt{coMod}_{(A,\Gamma)} \to \mathtt{coMod}_{(A,\Gamma)}.$$

We define $\mathrm{Ext}^i_{(A,\Gamma)}(M, -)$ to be the ith right derived functor of the functor above.

By the equivalence of categories between (A, Γ)-comodules and quasi-coherent sheaves over $\mathcal{M}_{(A,\Gamma)}$, this is the same as the right derived functors of gloabl sections over the quasi-coherent sheaf $\mathcal{F}_M$ associated to the comodule M. That is, these Ext groups are *sheaf cohomology over the stack* $\mathcal{M}_{(A,\Gamma)}$.

Let $t \in \mathbb{Z}$, and consider $\Sigma^t E_*$ as a comodule. Then we can consider

$$\mathrm{Hom}_{\mathtt{coMod}}(\Sigma^t E_*, -) : \mathtt{coMod}_{(E_*,E_*E)} \to \mathtt{coMod}_{(E_*,E_*E)}.$$

Let X be an arbitrary spectrum, so that E_*X is a comodule, and consider the right derived functors evaluated at E_*X. These are of the form

$$\mathrm{Ext}^s_{\mathtt{coMod}_{(E_*,E_*E)}}\left(\Sigma^t E_*, E_*X\right).$$

Theorem 4.8. (Adams, see [**Culver**]) Suppose that E is a flat ring spectrum, and let X be an arbitrary spectrum. Then there is a spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{E_*E}\left(\Sigma^s E_*, E_*X\right).$$

Under nice conditions, this converges to $\pi_*\left(X_E^\wedge\right)$, where $X_E^\wedge$ is the E-nilpotent completion of X.

So the study of the homotopy of any spectrum (locally) reduces to understanding the spectral sequence above. But even figuring out what E_*E is in most situations is a difficult result.

Example 4.9. When $E = H\mathbb{F}_p$, we have that

$$(H\mathbb{F}_p)_*H\mathbb{F}_p = \mathcal{A}^*,$$

called the *dual Steenrod algebra*. In general, we call E_*E the dual E-Steenrod algebra.

Let's try to use these ideas to compute the homotopy of MU and see where we get stuck

5. Computing $\pi_*\mathrm{MU}$

Recall:

Lemma 5.1. We have that

$$H_*(\mathrm{MU};\mathbb{Z}) = \mathbb{Z}[b_1, \ldots]$$

where $|b_i| = 2i$.

Lemma 5.2. We have that, rationally,

$$\pi_*\mathrm{MU} \otimes \mathbb{Q} \simeq H_*\mathrm{MU} \otimes \mathbb{Q}.$$

It suffices then to understand the p-torsion in $\pi_*\mathrm{MU}$, which is the same as understanding

$$\pi_*(\mathrm{MU}) \otimes \mathbb{Z}_p = \pi_*\left(\mathrm{MU}^\wedge_{(p)}\right)???$$

So remember our spectral sequence

$$E_2^{s,t} = \mathrm{Ext}^s_{E_*E}(\Sigma^s E_*, E_*X) \Rightarrow \pi_*\left(X^\wedge_E\right).$$

We want to apply it with $X = \mathrm{MU}$, and $E = H\mathbb{F}_p$. In this situation, we have

$$\mathrm{Ext}^{s,t}_{\mathcal{A}^*}(\mathbb{F}_p, (H\mathbb{F}_p)_*\mathrm{MU}) \Rightarrow \pi_*\mathrm{MU} \otimes \mathbb{Z}_p.$$

So we want to understand

$$(H\mathbb{F}_p)_*\mathrm{MU} = H_*(\mathrm{MU};\mathbb{F}_p)$$

as a comodule over the dual Steenrod algebra $\mathcal{A}^*$. In order to do this, we need to develop a bit more intuition for the category of comodules.

6. The box product for comodules

Let (A,Γ) be a Hopf algebroid, with Γ flat over A, and suppose that M and N are comodules, with coaction maps ψ_M and ψ_N respectively. Define the *cotensor product* of M and N as

$$M\square_\Gamma N := \ker\left(M\otimes_A N \xrightarrow{\psi_M\otimes N - M\otimes\psi_N} M\otimes_A\Gamma\otimes_A N\right).$$

This is *not* a comodule over the Hopf algebroid, it's not even an A-module! It's only a K-module, where K is our base ring. However, it allows us to understand the K-module structure on homs in $\mathtt{coMod}_{(A,\Gamma)}$ a little better.

Proposition 6.1. [Rav86, A.1.1.6] If M,N are left (A,Γ)-comodules, and M is projective over A, then we have that

$$\mathrm{Hom}_{(A,\Gamma)}(M,N) = \mathrm{Hom}_A(M,A)\square_\Gamma N.$$

Suppose that $f:(A,\Gamma)\to(B,\Sigma)$ is a map of Hopf algebroids (the definition is what you might expect it to be).

Lemma 6.2. We have that $\Gamma\otimes_A B$ is a right (B,Σ)-comodule, and we have a function

$$\begin{aligned}\mathtt{coMod}_{(A,\Gamma)} &\to \mathtt{coMod}_{(B,\Sigma)}\\ N &\mapsto (\Gamma\otimes_A B)\square_\Sigma N.\end{aligned}$$

Another perspective on this is that there is a *pullback-pushforward adjunction*

$$\begin{aligned}f^*:\mathtt{coMod}_{(A,\Gamma)} &\leftrightarrows \mathtt{coMod}_{(B,\Sigma)}:f_*\\ M &\mapsto B\otimes_A M\\ (\Gamma\otimes_A B)\square_\Sigma N &\leftarrow\!\shortmid N.\end{aligned}$$

Remark 6.3. If we stackify (A,Γ) and (B,Σ), these are exactly pullback and pushforward of quasi-coherent sheaves.

$$\mathtt{QCoh}(\mathcal{M}_{(A,\Gamma)}) \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathtt{QCoh}(\mathcal{M}_{(B,\Sigma)})$$

So we have a natural isomorphism

$$\mathrm{Hom}_{\mathtt{coMod}_{(B,\Sigma)}}(f^*M,N)\cong\mathrm{Hom}_{\mathtt{coMod}_{(A,\Gamma)}}(M,f_*N).$$

In particular we note that $f^*A=B$, so for any N, we have

$$\mathrm{Hom}_{\mathtt{coMod}_{(B,\Sigma)}}(B,N)\cong\mathrm{Hom}_{\mathtt{coMod}_{(A,\Gamma)}}(A,f_*N).$$

And by some homological algebra magic, this actually descends to an isomorphism on the derived functors of Hom:

$$\mathrm{Ext}^*_{(B,\Sigma)}(B, N) \cong \mathrm{Ext}^*_{(A,\Gamma)}(A, f_*N).$$

This type of argument is called a *change-of-rings* theorem. This one is due to Miller–Ravenel. We can use this one to finish computing the p-torsion in $\pi_*\mathrm{MU}$.

7. Setting up the spectral sequence for Quillen's theorem

Recall we want to understand

$$\mathrm{Ext}^{s,t}_{\mathcal{A}_*}(\mathbb{F}_p, H_*(\mathrm{MU}; \mathbb{F}_p)) \Rightarrow \pi_*(\mathrm{MU}) \otimes \mathbb{Z}_p.$$

So we need to understand the mod p homology of MU as a comodule over the dual Steenrod algebra $\mathcal{A}_*$.

We recall that $H\mathbb{F}_p$ is complex oriented — so there is a Thom class $u : \mathrm{MU} \to H\mathbb{F}_p$. Taking mod p homology, we get a map

$$H_*(\mathrm{MU}; \mathbb{F}_p) \to \mathcal{A}_*.$$

Notation 7.1. We denote by

$$P(\xi_1, \xi_2, \ldots)$$

a polynomial algebra over $\mathbb{F}_p$ on the generators ξ_i, in degree

$$|\xi_i| = \begin{cases} 2^n - 1 & p = 2 \\ 2(p^n - 1) & p > 2 \end{cases}$$

Define

$$P_* := \begin{cases} P(\xi_1^2, \xi_2^2, \ldots) & p = 2 \\ P(\xi_1, \xi_2, \ldots) & p > 2. \end{cases}$$

Lemma 7.2. [Rav86, p. 3.1.4] The image of the map $H_*(\mathrm{MU}; \mathbb{F}_p) \to \mathcal{A}_*$ is P_*.

Theorem 7.3. (Milnor, Novikov) We have that

$$H_*(\mathrm{MU}; \mathbb{F}_p) = P_* \otimes C,$$

as comodules over the dual Steenrod algebra, where $C = P(x_1, x_2, \ldots)$ for $|x_i| = 2i$, and i any integer which is not 1 less than a power of p.

Exercise 7.4. We have that

$$\mathcal{A}_* = P_* \otimes (\mathcal{A}_* \otimes_{P_*} \mathbb{F}_p).$$

Thus for any N we have

$$P_* \otimes N = \mathcal{A}_* \square_{\mathcal{A}_* \otimes_{P_*} \mathbb{F}_p} N.$$

In particular, we can rewrite

$$P_* \otimes C = \mathcal{A}_* \square_{\mathcal{A}_* \otimes_{P_*} \mathbb{F}_p} C.$$

We can now return to the computation at hand! With our new structure theorem about $H_*(\mathrm{MU}; \mathbb{F}_p)$, we see that

$$\mathrm{Ext}_{\mathcal{A}_*}\left(\mathbb{F}_p, H_*\left(\mathrm{MU}; \mathbb{F}_p\right)\right) = \mathrm{Ext}_{\mathcal{A}_*}\left(\mathbb{F}_p, \mathcal{A}_* \square_{\mathcal{A}_* \otimes_{P_*} \mathbb{F}_p} C\right).$$

Then we can apply the change-of-rings theorem! This tells us that

$$\mathrm{Ext}_{\mathcal{A}_*}\left(\mathbb{F}_p, \mathcal{A}_* \square_{\mathcal{A}_* \otimes_{P_*} \mathbb{F}_p} C\right) \simeq \mathrm{Ext}_{\mathcal{A}_* \otimes_{P_*} \mathbb{F}_p}(\mathbb{F}_p, C).$$

Finally for some magical reason(?) we are allowed to pull the C outside into a tensor.

$$\mathrm{Ext}_{\mathcal{A}_* \otimes_{P_*} \mathbb{F}_p}(\mathbb{F}_p, C) \simeq \mathrm{Ext}_{\mathcal{A}_* \otimes_{P_*} \mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p) \otimes C.$$

Lemma 7.5. [Rav86, p. 3.1.9] We have that

$$\mathrm{Ext}_{\mathcal{A}_* \otimes_{P_*} \mathbb{F}_p}(\mathbb{F}_p, \mathbb{F}_p) \simeq P(v_0, v_1, \ldots),$$

where v_i has bidegree $(1, 2p^i - 1)$.

Corollary 7.6. We have that

$$\mathrm{Ext}_{\mathcal{A}_*}\left(\mathbb{F}_p; H_*(\mathrm{MU}; \mathbb{F}_p)\right) = C \otimes P(v_0, v_1, \ldots).$$

Since $|v_i| = (1, 2p^i - 1)$, its total degree $(t - s)$ is always going to be even! In particular, there can be no nontrivial differentials, so the spectral sequence degenerates on the second page.

Corollary 7.7. We have that

$$\pi_*(\mathrm{MU}_p^\wedge) \cong \mathbb{Z}_p[v_0, v_1, \ldots] \otimes C.$$

8. Relative injectives

Definition 8.1. A *relative injective comodule* in $\mathtt{coMod}_{(A,\Gamma)}$ is a direct summand of an extended comodule.

Example 8.2. We say a spectrum K is a *relative E-injective* if K is a retract of a spectrum of the form $E \wedge X$ for some X. These are precisely the spectra whose E-homology (?) is relatively injective in $\mathtt{coMod}_{(E_*, E_*E)}$.

Lemma 8.3. [Goe04, p. 3.3] Suppose X is a spectrum whose homology E_*X is projective as an E_*-module. Then for all spectra Y there is a Hurewicz map

$$[X, Y] \to \mathtt{coMod}_{(E_*, E_*E)}(E_*X, E_*Y),$$

which will be an isomorphism if Y is a relative E-injective spectrum.

Definition 8.4. [Goe04, p. 3.4] An *E_*-Adams resolution for* a spectrum Y is a sequence

$$Y \to K^0 \to K^1 \to K^2 \to \cdots$$

so that each K^i is a relative E_*-injective, and for any other relative E_*-injective J, we have that the complex

$$\cdots \to [K^2, J] \to [K^1, J] \to [K^0, J] \to [Y, K] \to 0$$

is exact.

Under nice circumstances, we will have a spectral sequence

$$\mathrm{Ext}^s_{E_*E}\left(\Sigma^t E_* X, E_* Y\right) \Rightarrow \left[\Sigma^{t-s} X, L_E Y\right],$$

where L_E is the Bousfield localization of Y. When $X = S^0$, we have a spectral sequence converging to the E_*-local homotopy groups of Y.

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