THE HOMOTOPY COHERENT NERVE

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Abstract. Notes from an expository talk, given in the algebraic topology proseminar at UPenn, spring 2020.

References:

- Kerodon, Section 2.4
- nLab: the homotopy coherent nerve, Kan fibrant replacement
- MO202853, Zhen Lin
- MO324728, Why is Ex-infty useful
- Bergner, A model category structure on the category of simplicial categories
- Guillou - Kan's Ex functor

1. The homotopy coherent nerve

In the classical nerve construction, we recall that the nerve of a 1-category $\mathscr{C}$ is the simplicial set $N_\bullet\mathscr{C}$, whose n-cells $N_n\mathscr{C}$ are the strings of n-composable morphisms in $\mathscr{C}$.

We can adopt a slightly different perspective on this by viewing the ordered set $[n] = \{0 < 1 < \cdots < n\}$ as a category in itself, rather than as an object of the category $\mathbf{\Delta}$. We note then that we have a bijection

$$\{\text{order-preserving set maps } [n] \to [m]\} \longleftrightarrow \{\text{functors } [n] \to [m]\}.$$

We can then view $\mathbf{\Delta}$ as a subcategory of the 2-category $\mathtt{Cat}$, consisting of the categories $[n]$ for $n \geq 0$, and the functors between them.

Remark 1.1. Note that, for any category $\mathscr{C}$, the functor category $\mathrm{Fun}([n], \mathscr{C})$ is the collection of all n-composable morphisms in $\mathscr{C}$.

Definition 1.2. We can redefine the classical nerve of a category $\mathscr{C}$ as the simplicial set

$$\mathrm{Fun}(-, \mathscr{C}) : \mathbf{\Delta}^{\mathrm{op}} \to \mathtt{Set}.$$

A motivating example for the homotopy coherent nerve 1.3. Consider the simplicial set $N_\bullet\mathtt{Top}$, obtained by taking the nerve of the (small skeleton of the) category of

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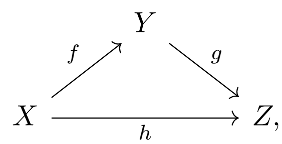
topological spaces. An element of $N_2\mathtt{Top}$ is a 2-cell which witnesses strict composition of spaces:



where here $h = g \circ f$. A question we might be motivated to ask is as follows: could we weaken the definition of the nerve so that it witnesses *weak composition*, instead of strict composition? What we mean by this is could we create an altered construction of the nerve where a 2-cell like that above would exist if $h \simeq g \circ f$ were *homotopic* instead of being strictly equal. If this were possible, it would make sense that we should have a 2-cell for *each* homotopy $H : h \simeq g \circ f$. This will be made explicit by the *homotopy coherent nerve*.

Definition 1.4. A *simplicially enriched category* (occasionally called a *simplicial category*, although this is overloaded terminology) $\mathscr{C}$ is a category enriched in `sSet`.

We will provide some examples below, but first give a glimpse of how this relates to ∞-categories.

Digression 1.5. Recall that an (n, r)-category is an n-category so that all k-morphisms are invertible for $k > r$. For example:

- a $(1, 0)$-category is just a groupoid
- an $(\infty, 1)$-category is generally called an ∞-category
- an $(\infty, 0)$-category is an ∞-groupoid.

If $\mathscr{C}$ is enriched in a category of (n, r)-categories, then $\mathscr{C}$ itself is an $(n+1, r+1)$-category. As a particular instance of this, if $\mathscr{C}$ were a simplicially enriched category, all of whose homs happened to be Kan complexes, then we might just as well think of $\mathscr{C}$ as being enriched in `Kan`. Since `Kan` is a category of $(\infty, 0)$-categories, then $\mathscr{C}$ is an $(\infty, 1)$-category. This hints at a more general notion that simplicially enriched categories might serve as a model for $(\infty, 1)$-categories. We will revisit this later.

Definition 1.6. A *simplicial functor* between simplicially enriched categories is a functor $F : \mathscr{C} \to \mathscr{D}$, which induces a map of simplicial sets for every pair of objects

$$\mathrm{Hom}_{\mathscr{C}}(X, Y)_\bullet \to \mathrm{Hom}_{\mathscr{D}}(FX, FY)_\bullet,$$

satisfying some natural conditions relating to composition. In particular, if $X = Y$, then it sends $\mathrm{id}_X \mapsto \mathrm{id}_{FX}$ under the map of 0-cells $\mathrm{Hom}_{\mathscr{C}}(X, X)_0 \to \mathrm{Hom}_{\mathscr{D}}(FX, FX)_0$.

Notation 1.7. We have a category whose objects are simplicial sets, and whose morphisms are simplicial functors. It is denoted (depending on who you are) by:

$$\begin{cases} \mathtt{Cat}_\Delta & \text{(Kerodon)} \\ \mathtt{sSet\text{-}Cat} & \text{(nLab).} \end{cases}$$

We will give an example of a simplicially enriched category.

Let $(Q, \leq)$ be a poset, and for any two elements $x, y \in Q$ with $x < y$ denote by $P_{x,y}$ the associated poset consisting of finite chains $\{x < x_0 < \cdots < x_n = y\}$ which begin at x and end at y. The order relation on $P_{x,y}$ is given by subdivision, e.g.

$$\{x < x_0 < y\} < \{x < x_0 < x_1 < y\}.$$

Definition 1.8. For any poset $(Q, \leq)$, we get a simplicial category $Path[Q]_\bullet$, defined by

- $\mathrm{ob} Path[Q]_\bullet = \mathrm{ob} Q$
- $\mathrm{Hom}_{Path[Q]_\bullet}(x, y) = N_\bullet P_{x,y}$ is the classical nerve of the poset $P_{x,y}$
- $\mathrm{id}_x \in \mathrm{Hom}_{Path[Q]_\bullet}(x, x)$
- Composition of 0-cells is given by taking a union; i.e.

$$\begin{gathered}\mathrm{Hom}_{Path[Q]_\bullet}(y, z)_0 \times \mathrm{Hom}_{Path[Q]_\bullet}(x, y)_0 \to \mathrm{Hom}_{Path[Q]_\bullet}(x, z)_0 \\ (\{x < x_1 < \cdots < x_n = y\}, \{y < y_1 < \cdots < y_m = z\}) \mapsto \{x < x_1 < \cdots < x_n = y < y_1 < \cdots < y_m = z\}.\end{gathered}$$

- Higher composition is given by an inclusion of the product of simplicial sets

$$\mathrm{Hom}_{Path[Q]_\bullet}(y, z) \times \mathrm{Hom}_{Path[Q]_\bullet}(x, y) \to \mathrm{Hom}_{Path[Q]_\bullet}(x, z).$$

  This will become clear in examples.

Example 1.9. If $Q = [1] = \{0 < 1\}$, then $Path[1]_\bullet$ consists of the two objects 0 and 1, and

$$\mathrm{Hom}_{Path[1]_\bullet}(0, 1) = N_\bullet(P_{0,1}) = \Delta^0.$$

Example 1.10. If $Q = [2]$, then $Path[2]_\bullet$ has three objects, 0, 1, and 2. We know that $\mathrm{Hom}_{Path[2]_\bullet}(0, 1)$ and $\mathrm{Hom}_{Path[2]_\bullet}(1, 2)$ are 0-simplices by the previous example. We check that $\mathrm{Hom}_{Path[2]_\bullet}(0, 2) = N_\bullet P_{0,2}$ is the nerve of the poset $\{0 < 2\} \to \{0 < 1 < 2\}$, and hence is a 1-simplex Δ^1. We think about the composition law as a disjoint union of simplicial sets

$$\begin{gathered}\mathrm{Hom}_{Path[2]_\bullet}(1, 2) \times \mathrm{Hom}_{Path[2]_\bullet}(0, 1) \to \mathrm{Hom}_{Path[2]_\bullet}(0, 2) \\ \Delta^0 \times \Delta^0 \hookrightarrow \Delta^1 \\ (\{1 < 2\}, \{0 < 1\}) \mapsto \{0 < 1 < 2\}.\end{gathered}$$

Thus composition sends the product of the two 0-simplices (that is, a 0-simplex) to the 0-simplex $\{0 < 1 < 2\} \in \mathrm{Hom}_{Path[2]_\bullet}(0, 2)_0$.

Example 1.11. If $Q = [3]$, then we have four objects. Again we know almost all the homs by the previous examples, but we compute that

$$\begin{aligned}\mathrm{Hom}_{Path[3]_\bullet}(0,3) = N_\bullet P_{0,3} &= N_\bullet \left(\begin{array}{ccc} \{0<3\} & \longrightarrow & \{0<1<3\} \\ \downarrow & \searrow & \downarrow \\ \{0<2<3\} & \longrightarrow & \{0<1<2<3\} \end{array} \right) \\ &= (\Delta^1)^2. \end{aligned}$$

There are three nontrivial compositions we can make:

$$\begin{aligned}
&\mathrm{Hom}_{Path[3]_\bullet}(1,2) \times \mathrm{Hom}_{Path[3]_\bullet}(0,1) \to \mathrm{Hom}_{Path[3]_\bullet}(0,2) \\
&\mathrm{Hom}_{Path[3]_\bullet}(1,3) \times \mathrm{Hom}_{Path[3]_\bullet}(0,1) \to \mathrm{Hom}_{Path[3]_\bullet}(0,3) \\
&\mathrm{Hom}_{Path[3]_\bullet}(2,3) \times \mathrm{Hom}_{Path[3]_\bullet}(0,2) \to \mathrm{Hom}_{Path[3]_\bullet}(0,3).
\end{aligned}$$

The first we have already seen, and the second two should be symmetric in some sense, so without loss of generality let's pick the last one to study.

Call $X = \mathrm{Hom}_{Path[3]_\bullet}(2,3) \times \mathrm{Hom}_{Path[3]_\bullet}(0,2)$ the product of these two simplicial sets. We recall that the n-cells of a product of simplicial sets are given by the n-cells of each term, thus we see that X has two 0-cells:

$$\begin{aligned} X_0 = \{ &(\{2<3\}, \{0<2\}) \\ &(\{2<3\}, \{0<1<2\}) \} \end{aligned}$$

Moreover the edge $\{0<2\} \to \{0<1<2\}$ in $\mathrm{Hom}_{Path[3]_\bullet}(0,2)_1$ induces a map between these cells in X_1:

$$(\{2<3\}, \{0<2\}) \to (\{2<3\}, \{0<1<2\}).$$

Thus X looks like a 1-simplex Δ^1. If we now look at the image of X in $\mathrm{Hom}_{Path[3]_\bullet}(0,3)$, we see that the 0-cells are taken to their unions, and the edge between them is preserved, thus we obtain the lower edge of $\mathrm{Hom}_{Path[3]_\bullet}(0,3)$:

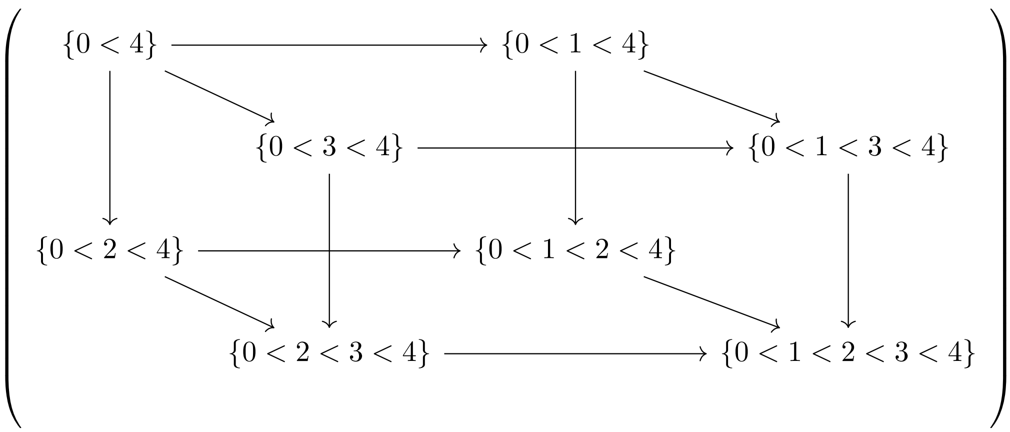
$$\begin{aligned} &\mathrm{Hom}_{Path[3]_\bullet}(2,3) \times \mathrm{Hom}_{Path[3]_\bullet}(0,2) \to \mathrm{Hom}_{Path[3]_\bullet}(0,3) \\ &\qquad \Delta^1 \hookrightarrow N_\bullet P_{0,3} = N_\bullet \left(\begin{array}{ccc} \{0<3\} & \longrightarrow & \{0<1<3\} \\ \downarrow & \searrow & \downarrow \\ \{0<2<3\} & \longrightarrow & \{0<1<2<3\} \end{array} \right) \end{aligned}$$

Example 1.12. For one more nontrivial example, we look at the composition

$$\mathrm{Hom}_{Path[4]_\bullet}(2,4) \times \mathrm{Hom}_{Path[4]_\bullet}(0,2) \to \mathrm{Hom}_{Path[4]_\bullet}(0,4).$$

One may check first of all that $\mathrm{Hom}_{Path[4]_\bullet}(0,4)$ is the following simplicial set:

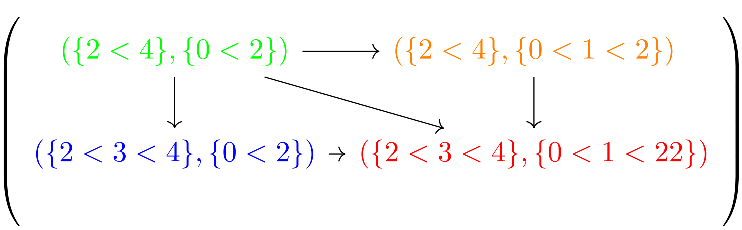
$\mathrm{Hom}_{Path[4]_\bullet}(0,4) = N_\bullet$



along with some additional composite arrows that are not drawn for the ease of the reader.

We have four 0-simplices in the product $\mathrm{Hom}_{Path[4]_\bullet}(2,4) \times \mathrm{Hom}_{Path[4]_\bullet}(0,2)$, and together with their edges and faces, form a square:

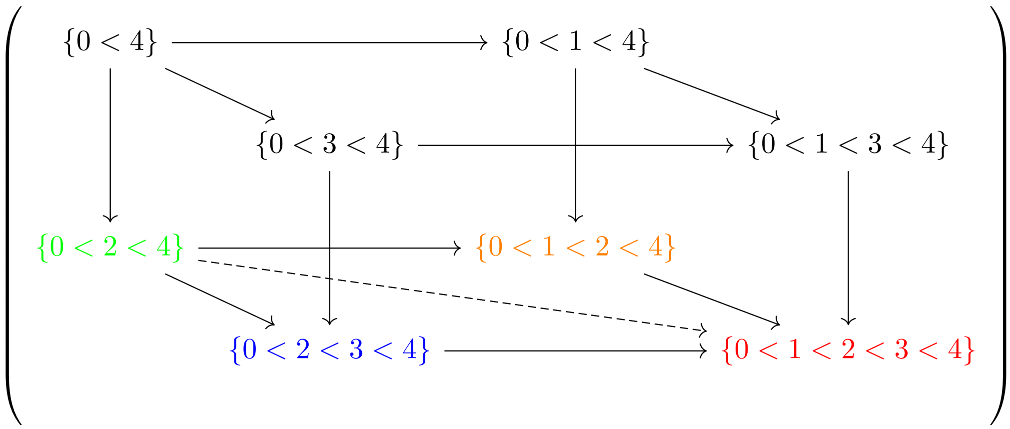
$N_\bullet$



$\cong (\Delta^1)^2$

The composition $\mathrm{Hom}_{Path[4]_\bullet}(2,4) \times \mathrm{Hom}_{Path[4]_\bullet}(0,2) \to \mathrm{Hom}_{Path[4]_\bullet}(0,4)$ includes this into the cube as the bottom face:

$(\Delta^1)^2 \hookrightarrow N_\bullet$



Example 1.13. In general for $[n]$, one has that $N_\bullet P_{i,j} = (\Delta^1)^{j-i-1}$.

We note that this path construction defines a functor

$$Path[-]_\bullet : \mathtt{Poset} \to \mathtt{Cat}_\Delta.$$

Remark 1.14. For every $x, y \in Path[Q]_0$, we have that

$$\mathrm{Hom}_{Path[Q]_\bullet}(x,y) \simeq *$$

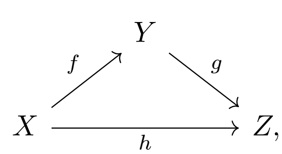
is contractible as a simplicial set (since it has an initial object $\{x < y\}$). Thus we should think of $Path[Q]_\bullet$ as some sort of "categorical thickening" of Q.

Definition 1.15. For any simplicially enriched category $\mathscr{C} \in \mathtt{Cat}_\Delta$, we define its *homotopy coherent nerve* $N_\bullet^{\mathrm{hc}}(\mathscr{C})$ as the simplicial set

$$\begin{aligned} \mathbf{\Delta}^{\mathrm{op}} &\to \mathtt{Set} \\ [n] &\mapsto \mathrm{Hom}_{\mathtt{Cat}_\Delta}(Path[n]_\bullet, \mathscr{C}). \end{aligned}$$

Remark 1.16.

1. (1) The vertices of $N_\bullet^{\mathrm{hc}}(\mathscr{C})$ are the objects of $\mathscr{C}$
2. (2) The edges of $N_\bullet^{\mathrm{hc}}(\mathscr{C})$ are the morphisms of $\mathscr{C}$
3. (3) The face maps $d_1, d_0 : N_1^{\mathrm{hc}}(\mathscr{C}) \to N_0^{\mathrm{hc}}(\mathscr{C})$ are the pair $(\mathrm{codom}, \mathrm{dom})$
4. (4) The 0th degeneracy map $s_0 : N_0^{\mathrm{hc}}(\mathscr{C}) \to N_1^{\mathrm{hc}}(\mathscr{C})$ sends x to id_x.
5. (5) An element of $N_2^{\mathrm{hc}}(\mathscr{C})$ is a map $Path[2]_\bullet \to \mathscr{C}$, which is the data of a (commutative or non-commutative) diagram in $\mathscr{C}$



and a homotopy $h \Rightarrow g \circ f$, which corresponds to the image of the edge $\{0,2\} \to \{0 < 1 < 2\}$ in the simplical set $Path[2]_\bullet$.

One can prove that 2-cells correspond bijectively to homotopies at the simplicial level (see Kerodon, 2.4.6).

Definition 1.17. There is a functor

$$\begin{aligned}(-)_0 : \mathtt{Cat}_\Delta &\to \mathtt{Cat} \\ \mathscr{C} &\mapsto \mathscr{C}_0,\end{aligned}$$

given by taking the 0-cells of each of the simplicial homs. This is called the *underlying 1-category* of $\mathscr{C}$. Similarly, we have a functor in the reverse direction

$$\begin{aligned}\underline{(-)} : \mathtt{Cat} &\to \mathtt{Cat}_\Delta \\ \mathscr{D} &\mapsto \underline{\mathscr{D}},\end{aligned}$$

by sending a 1-category to the *constant simplicial category* $\underline{\mathscr{D}}$, whose homs are just disjoint unions of 0-simplices, corresponding to elements in mor$\mathscr{D}$. We claim that there is an adjunction of categories

$$\mathrm{Hom}_{\mathtt{Cat}}\left(\mathscr{C}, \mathscr{D}_0\right) = \mathrm{Hom}_{\mathtt{Cat}_\Delta}\left(\underline{\mathscr{C}}, \mathscr{D}\right).$$

Remark 1.18. As hinted at earlier, we always have a simplicial functor $Path[Q]_\bullet \to Q$, simply given by contracting homs to a point. We note that for any ordinary functor of categories $Q \to \mathscr{C}_0$, we can precompose with the collapse map and post-compose with the inclusion of the 0-truncation to obtain a composite

$$Path[Q]_\bullet \to Q \to \mathscr{C}_0 \to \mathscr{C}.$$

This induces an inclusion

$$\mathrm{Fun}_{\mathtt{Cat}}(Q, \mathscr{C}_0) \hookrightarrow \mathrm{Fun}_{\mathtt{Cat}_\Delta}(Path[Q]_\bullet, \mathscr{C}).$$

Restricting our attention purely to elements of $\mathbf{\Delta}$, we get an inclusion

$$\mathrm{Fun}_{\mathtt{Cat}}([n], \mathscr{C}_0) \hookrightarrow \mathrm{Fun}_{\mathtt{Cat}_\Delta}(Path[n]_\bullet, \mathscr{C}),$$

which induces an embedding of simplicial sets

$$N_\bullet(\mathscr{C}) \hookrightarrow N_\bullet^{\mathrm{hc}}(\mathscr{C}).$$

This is a bijection on edges and vertices, as we might expect. In general, we should *not* expect this to be an isomorphism. However we do have that

$$N_\bullet(\mathscr{C}) \xrightarrow{\cong} N_\bullet^{\mathrm{hc}}\left(\underline{\mathscr{C}}\right)$$

is an isomorphism of simplicial sets.

2. Homotopy coherent nerve is a right adjoint

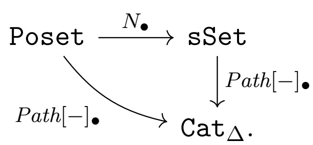
Definition 2.1. The classical nerve fits into an adjunction

$$h : \mathtt{sSet} \rightleftarrows \mathtt{Cat} : N_\bullet.$$

Here h is the *homotopy category* of a simplicial set. We have that $\mathrm{ob}(hX) := X_0$, and that $\mathrm{mor}(hX)$ is freely generated by elements of X_1, equipped with their canonical direction as simplicial edges, modulo composition relations witnessed by elements of X_2.

One may suspect that an analogous statement holds for the homotopy coherent nerve.

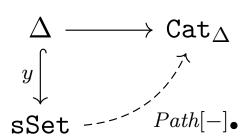
Definition 2.2. We may extend the functor $Path[-]_\bullet : \mathtt{Poset} \to \mathtt{Cat}_\Delta$ to a functor $Path[-]_\bullet : \mathtt{sSet} \to \mathtt{Cat}_\Delta$ valued on all simplicial sets, fitting into the diagram below



This is defined by

$$\begin{aligned} Path[-]_\bullet : \mathtt{sSet} &\to \mathtt{Cat}_\Delta \\ X &\mapsto \int^{[n]\in\mathbf{\Delta}} X_n \cdot Path[n]_\bullet. \end{aligned}$$

That is, it is the left Kan extension



By abuse of notation, we refer to $Path[n]_\bullet$ both as the path simplicial category of the poset $[n]$, and $Path[-]_\bullet$ applied to the simplicial set Δ^n.

Theorem 2.3. There is an adjunction

$$Path[-]_\bullet : \mathtt{sSet} \rightleftarrows \mathtt{Cat}_\Delta : N_\bullet^{\mathrm{hc}}.$$

Reality check 2.4. This aligns with our intuition, since

$$\mathrm{Hom}_{\mathtt{Cat}_\Delta}(Path[n]_\bullet, \mathscr{C}) \cong \mathrm{Hom}_{\mathtt{sSet}}(\Delta^n, N_\bullet^{\mathrm{hc}}(\mathscr{C})) = N_n^{\mathrm{hc}}(\mathscr{C}).$$

That is, $Path[n]_\bullet$ corepresents n-cells in the homotopy coherent nerve, as we already knew.

3. Kan locality

Definition 3.1. For a simplex $\Delta^k \in \texttt{sSet}$, we define its *barycentric subdivision*, denoted $\mathrm{sd}\Delta^k$, to be the nerve of the poset of non-degenerate sub-simplices. This is intimately related to our definitions of the nerve of $P_{x,y}$ before, however we drop the restriction that our posets start and end with x and y. Here, for example, we have that

$$\mathrm{sd}\Delta^1 = \{\{0\} \to \{0,1\} \leftarrow \{1\}\}$$

$$\mathrm{sd}\Delta^2 = \left\{ \begin{array}{ccccc} \{0\} & \longrightarrow & \{0,1\} & \longleftarrow & \{1\} \\ & \searrow \quad \searrow & \downarrow & \swarrow \quad \swarrow & \\ \{0,2\} & \longrightarrow & \{0,1,2\} & \longleftarrow & \{1,2\} \\ & \nwarrow & \uparrow & \nearrow & \\ & & \{2\} & & \end{array} \right\}$$

We define a functor

$$\mathrm{Ex} : \texttt{sSet} \to \texttt{sSet},$$

where

$$(\mathrm{Ex}X)_k := \mathrm{Hom}_{\texttt{sSet}}(\mathrm{sd}\Delta^k, X).$$

Remark 3.2. For any simplicial set, there is a functor

$$\mathrm{Ex}^\infty : \texttt{sSet} \to \texttt{sSet},$$

defined to be the colimit

$$\mathrm{Ex}^\infty(X) := \mathrm{colim}\left(X \to \mathrm{Ex}X \to \mathrm{Ex}(\mathrm{Ex}X) \to \cdots\right).$$

Properties of $\mathbf{Ex}^\infty$ 3.3. This functor has the following properties (among many others)

(1) Since $\mathrm{sd}\Delta^0 \cong \Delta^0$, one sees Ex (and thus Ex^∞) preserves 0-simplices
(2) For any X, we have that $\mathrm{Ex}^\infty X$ is a Kan fibration
(3) We have that Ex^∞ is a fibrant replacement functor in the standard (Kan-Quillen) model structure on `sSet`[1]
(4) Ex^∞ preserves finite products, finite limits, filtered colimits, fibrations and acyclic fibrations, weak equivalences.

Remark 3.4. The functor $\mathrm{Sing}_\bullet|-| : \texttt{sSet} \to \texttt{sSet}$ is also a fibrant replacement functor, is easier to describe than Ex^∞, and preserves fibrations and finite limits. However, it does not preserve 0-simplices, and in general $\mathrm{Sing}_\bullet|X|$ is much bigger than $\mathrm{Ex}^\infty(X)$. Moreover,

[1] Cofibrations are levelwise injections, weak equivalences are those whose geom. realization is a weak equivalence of spaces, fibrations are Kan fibrations, everything is cofibrant, fibrant objects are precisely the Kan complexes.

Ex^∞ does not require us to use the category of spaces, so in that sense it is more general; the definition is internal to $\mathtt{sSet}$. See Guillou - Kan's Ex functor

Definition 3.5. We say $\mathscr{C} \in \mathtt{Cat}_\Delta$ is *locally Kan* if, for all $x, y \in \mathscr{C}$, we have that

$$\mathrm{Hom}_\mathscr{C}(x,y) \in \mathtt{Kan} \subseteq \mathtt{sSet}.$$

Theorem 3.6. *(Cordier-Porter)* If $\mathscr{C}$ is a locally Kan simplicially enriched category, then $N_\bullet^{\mathrm{hc}}(\mathscr{C})$ is an $(\infty,1)$-category.

Corollary 3.7. For any $\mathscr{C} \in \mathtt{Cat}_\Delta$, we can obtain a locally Kan simplicially enriched category, denoted $\mathrm{Ex}^\infty\mathscr{C}$, by applying the functor Ex^∞ at every hom.

Definition 3.8. There is a functor

$$\pi_0 : \mathtt{sSet} \to \mathtt{Set},$$

defined by

$$\pi_0(X) := \mathrm{coeq}\left(d_1, d_0 : X_1 \rightrightarrows X_0\right).$$

That is, it is the *set of connected components* of X. If X is a Kan complex, then the image $d_1, d_0 : X_1 \to X_0 \times X_0$ is an equivalence relation on X_0, so we can quotient out by it to get the coequalizer.

Definition 3.9. For any $\mathscr{C} \in \mathtt{Cat}_\Delta$, define by $\pi_0\mathscr{C}$ the *category of components*, given by taking π_0 on every hom-sset in $\mathscr{C}$; that is

$$\mathrm{Hom}_{\pi_0\mathscr{C}}(a,b) := \pi_0\mathrm{Hom}_\mathscr{C}(a,b).$$

We say $g \in \mathrm{Hom}_\mathscr{C}(a,b)_0$ is a *homotopy equivalence* if g becomes an isomorphism in $\pi_0\mathscr{C}$.

Proposition 3.10. Let $\mathscr{C}$ be any simplicially enriched category. Then we have that

$$N_\bullet^{\mathrm{hc}}\left(\mathrm{Ex}^\infty X\right)$$

is an $(\infty,1)$-category whose homotopy category is isomorphic to $\pi_0\mathscr{C}$.

4. The Bergner model structure on $\mathtt{Cat}_\Delta$

Definition 4.1. We say that a simplicial functor $F : \mathscr{C}$ is a *Dwyer-Kan weak equivalence* if

(1) for any $a, b \in \mathscr{C}$, the induced map

$$\mathrm{Hom}_\mathscr{C}(a,b) \to \mathrm{Hom}_\mathscr{D}(fa, fb)$$

is a weak equivalence of simplicial sets (in the standard model structure)

(2) the induced functor $\pi_0 F : \pi_0\mathscr{C} \to \pi_0\mathscr{D}$ is an equivalence of 1-categories

Definition 4.2. We say $F : \mathscr{C} \to \mathscr{D}$ is a *fibration* if

(1) for any $a, b \in \mathscr{C}$, the map

$$\mathrm{Hom}_{\mathscr{C}}(a, b) \to \mathrm{Hom}_{\mathscr{D}}(fa, fb)$$

is a (Kan) fibration of ssets

(2) for any $c \in \mathscr{C}$ and $d \in \mathscr{D}$, and homotopy equivalence $g \in \mathrm{Hom}_{\mathscr{D}}(fc, d)_0$, there is an object $c' \in \mathscr{C}$ and homotopy equivalence $h : c \to c'$ which fits into the (strictly) commutative diagram

$$\begin{array}{ccc} c & \xrightarrow{h} & c' \\ & \searrow & \downarrow \\ & & d \end{array}$$

concisely; these are the maps that induce isofibrations on π_0.

Theorem 4.3. There is a (right proper, cofibrantly generated model) category on $\mathtt{Cat}_\Delta$ with weak equivalences given by the Dwyer-Kan weak equivalences, and the prescribed fibrations above.

The fibrant objects in this structure are exactly those categories enriched in $\mathtt{Kan}$.

4.1. **The Joyal model structure on ssets.**

Definition 4.4. The *Joyal model structure on simplicial sets* is given by

- cofibrations are monomorphisms
- weak equivalences are those maps of simplicial sets $f : X \to Y$ so that the induced simplicial functor

$$Path[f]_\bullet : Path[X]_\bullet \to Path[Y]_\bullet$$

is a Dwyer-Kan weak equivalence of simplicially enriched categories.

Proposition 4.5. The fibrant objects in $\mathtt{sSet}_{\mathrm{Joyal}}$ are precisely the quasi-categories.

It almost looks as though this model structure was designed intentionally to interact well with the Bergner model structure. Indeed this is the case.

Theorem 4.6. The adjunction

$$Path[-]_\bullet : \mathtt{sSet}_{\mathrm{Joyal}} \rightleftarrows \mathtt{Cat}_\Delta : N_\bullet^{\mathrm{hc}}$$

is a Quillen equivalence.

Corollary 4.7. The right adjoint preserves fibrant objects, thus the homotopy coherent nerve of a $\mathtt{Kan}$-enriched category is a quasi-category.

This allows us to translate between two models of $(\infty, 1)$-categories.