The Generalized Poincaré Conjecture Using s-Cobordism

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In the study of manifolds, one generally cares about the classification of manifolds up to diffeomorphism or homeomorphism. In algebraic topology, we often work with an even weaker notion of equivalence, called *homotopy equivalence*. The interval $I = [0,1]$ allows us to form parametrized continuous changes in topological spaces. This is the basic idea behind homotopy theory.

Definition 1. Let $f : X \to Y$ and $g : X \to Y$ be continuous maps between topological spaces. We say that f and g are *homotopic*, denoted $f \simeq g$, if there exists a continuous map

$$H : X \times I \to Y,$$

such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. In this case we call H a *homotopy* between f and g.

We can check that homotopy forms an equivalence relation on the class of maps $X \to Y$.

Definition 2. Two spaces X and Y are *homotopy equivalent*, denoted $X \simeq Y$ if there exist maps $f : X \to Y$, and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$.

Every homeomorphism is a homotopy equivalence, but not the other way around.

Example 1. The closed unit disk D^n is homotopy equivalent to a point $*$, but they are clearly not homeomorphic.

In general, spaces which are homotopy equivalent to a point are called *contractible*.

The nth homotopy group $\pi_n(X, x_0)$ is the set of all "based" homotopy classes of maps $S^n \to X$ sending $1 \mapsto x_0$, where a based homotopy $H : S^n \times I \to X$ sending $(1, t) \mapsto x_0$ for all t. This is a group if $n \geq 1$, and an abelian group if $n \geq 2$.

In particular, $\pi_0(X, x_0)$ is just the set of path components of X.

Proposition 1. If $f : X \to Y$ is a homotopy equivalence (with inverse $g : Y \to X$), it induces isomorphisms on each $\pi_n(X, x_0) \cong \pi_n(Y, f(x_0))$.

Proof. Pick a basepoint $x_0 \in X$. Since $gf \simeq \mathrm{id}_X$, we have that x_0 and $g \circ f(x_0)$ lie in the same path component of X. It is easy to see then that $\pi_n(X, x_0) \cong \pi_n(X, (g \circ f)(x_0))$. Then consider the composition of maps induced by post-composition

$$\pi_n(X, x_0) \xrightarrow{f_*} \pi_n(Y, f(x_0)) \xrightarrow{g_*} \pi_n(X, g \circ f(x_0)).$$

This is an isomorphism since $g \circ f \simeq \mathrm{id}_X$. Similarly, we can consider the composition $f_* \circ g_*$ which is a group isomorphism, and together we conclude that $\pi_n(X, x_0) \cong \pi_n(Y, f(x_0))$. $\square$

We will call an n-manifold M a *homotopy n-sphere* if it is homotopy equivalent to S^n.

Theorem 2. (The Generalized Poincaré Conjecture) Every homotopy n-sphere is homeomorphic to S^n.

A homotopy equivalence induces a bijection on path components, so we can just worry about connected manifolds.

1. The Case $n = 1$

The following is a classical fact.

Theorem 3. (Classification of 1-manifolds) Every connected 1-manifold is homeomorphic to one of the following:

1. the real line $\mathbb{R}$

2. a ray $\mathbb{R}_{\geq 0}$

3. the circle S^1

4. the unit interval $[0, 1]$.

Therefore, every connected 1-manifold is either contractible, or homeomorphic to S^1.

2. The Case $n = 2$

The classification of surfaces is a relatively easy theorem, often proved at the end of an introductory algebraic topology course.

Definition 3. The *connected sum $X \# Y$* of two n-manifolds X and Y is obtained by cutting out an open disk D^n from each of X and Y, and then gluing X and Y together along the boundaries $\partial D^n = S^{n-1}$.

It is not difficult to see that homeomorphism classes of closed n-manifolds form a monoid under #, with identity element S^n.

Theorem 4. (Classification of closed surfaces) Any connected closed surface is homeomorphic to one of
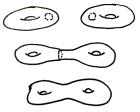
1. the sphere S^2

2. a connected sum of tori $\#^g T^2$, for $g \geq 1$

Figure 1: The connected sum of two tori

3. a connected sum of real projective planes $\#^k \mathbb{RP}^2$, for $k \geq 1$.

One can show that

$$\pi_1(\#^g T^2) = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$
$$\pi_1(\#^k \mathbb{RP}^2) = \langle a_1, \ldots, a_k \ : \ a_1^2 \cdots a_k^2 = 1 \rangle.$$

Neither of these groups are trivial for any choice of g or k. Thus if $M \simeq S^2$, it must have fundamental group $\pi_1(M) = \pi_1(S_2) = \{e\}$. Therefore $M \cong S^2$.

(Another way to see this is that the universal covers of $\#^g T^2$ and $\#^k \mathbb{RP}^2$ are contractible, and thus have trivial higher homotopy groups, since $\pi_i(M) \cong \pi_i(\widetilde{M})$ for $i \geq 2$. But $\pi_2(S^2) \cong \mathbb{Z}$.)

3. The Case $n = 3$

Perelman proved this in 2003 (and won the Fields Medal in 2006) for resolving this case. It involves the theory of surgery on manifolds.

4. The Case $n = 4$

Freedman solved this in 1982, and also received a Fields Medal.

5. The Case $n \geq 5$

In order to prove the Generalized Poincaré conjecture in high dimensions, we must first discuss some of the theory of cobordisms.

Definition 4. Let M and N be two n-manifolds. We say they are *cobordant* if there exists a compact $n + 1$-dimensional manifold W such that $\partial W = M \amalg N$. In this case, we call W a *cobordism* between M and N.

Definition 5. Let W and W' be cobordisms over M. We say that they are *isomorphic* if there exists a homeomorphism $f : W \xrightarrow{\cong} W'$ which restricts to the identity map on M.

Example 2.

1. Any manifold M is cobordant to itself, via $W = M \times [0, 1]$.

2. If $M = \partial W$ for some $n + 1$-dimensional manifold W, then M is cobordant to the empty set, by simply noticing that $\partial W = M \amalg \varnothing$.

3. If $M = \partial W_1$ and $N = \partial W_2$, then M and N are cobordant by taking the disjoint union $W = W_1 \amalg W_2$.

4. For any two n-manifolds M and N, we have that $M \# N$ and $M \amalg N$ are cobordant. This generalizes the "pair of pants" cobordism in Figure 2.

5. Any two manifolds with the same Stiefel-Whitney numbers are cobordant.

Figure 2: The "pair of pants" exhibiting a cobordism between S^1 and $S^1 \amalg S^1$

By $\partial W = M \amalg N$, we really mean that $\partial W = i(M) \amalg j(N)$, where $i : M \hookrightarrow \partial W$ and $j : N \hookrightarrow \partial W$ are embeddings. However, we could strengthen this condition a little bit:

Definition 6. Let M and N be two manifolds, and let W be a cobordism between them. We say that W is an *h-cobordism* if the inclusion maps $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences.

We may think of h-cobordisms as witnesses to a homotopy equivalence between M and N.

Example 3. For any manifold M, we always have the trivial h-cobordism $M \times [0, 1]$, since the inclusion $M \hookrightarrow M \times [0, 1]$ is a homotopy equivalence.

As an abrupt change of pace which will become clear in a minute, we will shift gears and talk about K-theory.

5.1. K_1 and The Whitehead Group

The algebraic K-groups are a series of functors $K_n : \textbf{Ring} \to \textbf{Grp}$, which have ties to many fields of mathematics, including number theory, intersection theory, and motivic homotopy theory. Algebraic K-theory is intimately related to topological K-theory, which is used to classify vector bundles over spaces.

The algebraic K-groups are typically exceedingly difficult to compute. Even $K_n(\mathbb{Z})$ is not know for all $n \in \mathcal{N}$, and the statement that $K_{4n}(\mathbb{Z}) = 0$ for all n is equivalent to the Vandiver conjecture about the class number of the maximal real subfield of a cyclotomic field.

The first K-group K_1 is surprisingly easy to describe. For a ring R, we have inclusions $\text{GL}_n(R) \hookrightarrow \text{GL}_{n+1}(R)$ via

$$ A \longmapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}. $$

Taking the union (really the colimit) over all n, we obtain the *infinite general linear group* $\text{GL}(R)$. Dividing out by its commutator, we obtain the first K-group:

$$ K_1(R) := \frac{\text{GL}(R)}{[\text{GL}(R), \text{GL}(R)]}. $$

In the case where R is a Euclidean domain (in particular a field), we have that $K_1(R) = R^\times$. For a proof of this, see Chapter III of Weibel's K-book.

Given a group G, and group ring $\mathbb{Z}[G]$, we obtain a canonical inclusion

$$ G \times \{\pm 1\} \to K_1(\mathbb{Z}[G]) $$
$$ (g, \pm 1) \mapsto (\pm g) \in \text{GL}_1(\mathbb{Z}[G]) \subseteq K_1(\mathbb{Z}[G]). $$

The cokernel of this map is defined to be the *Whitehead group of G*

$$ \text{Wh}(G) := \text{coker}\left(G \times \{\pm 1\} \to K_1(\mathbb{Z}[G]) \right). $$

Example 4. The Whitehead group of the trivial group $\{e\}$ is the cokernel of the map

$$ \{e\} \times \{\pm 1\} \to K_1(\mathbb{Z}[\{e\}]) = K_1(\mathbb{Z}) = \mathbb{Z}^\times = \{\pm 1\}, $$

which is trivial. So $\text{Wh}(\{e\}) = \{e\}$.

Any other examples involve extremely difficult computations, but luckily, this will be the only Whitehead group we need.

5.2. The s-Cobordism Theorem

Here is where our digression will make sense.

Theorem 5. (The s-cobordism theorem) For $n \geq 4$, let M be a connected, closed n-manifold. Then there is a bijection

$$\mathrm{Wh}(\pi_1(G)) \longleftrightarrow \{\text{isomorphism classes of } h\text{-cobordisms over } M\}.$$

We will conclude by proving the Poincaré conjecture in high dimensions.

Theorem 6. (The Generalized Poincaré Conjecture in Dimension ≥ 5) Let M be a closed manifold of dimension $n \geq 5$, such that M is homotopy equivalent to S^n. Then M is homeomorphic to S^n.

Proof. Since M is a manifold, we may pick two distinct points contained in closed n-dimensional disjoint disks D_0 and D_1, respectively. We define $W = M \smallsetminus \mathrm{int}(D_0 \cup D_1)$. This is a closed n-manifold with boundary given by

$$\partial W = \partial D_0 \cup \partial D_1 = S_0^{n-1} \cup S_1^{n-1},$$

which is the disjoint union of two $(n-1)$-spheres. Thus W defines a cobordism between S_0^{n-1} and S_1^{n-1}.

Since M and S^n are homotopy equivalent, we see that W is homotopy equivalent to $S^n \smallsetminus \mathrm{int}(D_0 \cup D_1) \cong S^{n-1} \times [0,1]$.

Therefore the inclusion maps $S_0^{n-1} \hookrightarrow W$ and $S_1^{n-1} \hookrightarrow W$ are homotopy equivalences, and we may see that W is an h-cobordism over S^{n-1}.

Since n is large enough, we have that $\pi_1(S^{n-1}) = \{e\}$, and thus that $\mathrm{Wh}(\pi_1(S^{n-1})) = \mathrm{Wh}(\{e\}) = \{e\}$. By the h-cobordism theorem, there is one isomorphism class of h-cobordisms over S^{n-1}, namely the class given by the cylinder $S^{n-1} \times [0,1]$.

We then have that there exists a homeomorphism $f : S^{n-1} \times [0,1] \to W$ such that $f\big|_{S^{n-1} \times \{0\}} = \mathrm{id}_{S^{n-1}}$.

By the Alexander trick, any homeomorphism on S^{n-1} can be extended to a homeomorphism $D^n \to D^n$. So we may glue back the top and bottom of the cylinder, to obtain a homeomorphism $S^n \xrightarrow{\cong} M$.

Thus we have proven the generalized Poincaré conjecture in dimensions $n \geq 4$. $\qquad \square$

6. Other categories to consider

Instead of working in *Top*, we can also consider the categories PL^1, the category whose objects are piecewise linear manifolds (the transition maps are piecewise linear), or *Diff*, whose objects are differentiable manifolds.

We then can ask: if M is a homotopy n-sphere in *PL* (resp. *Diff*), is it PL-isomorphic (resp. diffeomorphic) to S^n?

We then obtain the following status of the generalized Poincaré Conjecture:

Dimension	*Top*	*PL*	*Diff*
$n = 1, 2, 3^2$	True	True	True
$n = 4$	True	?	?
$n = 5, 6$	True	True	True
$n = 7$	True	True	False
$n \geq 8$	True	True	generally false but sometimes true

This last case, for differentiable manifolds, is particularly strange. Let's combine our results to see what's going on.

Let M be a differentiable n-manifold which is homotopy equivalent to S^n. The generalized conjecture in *Diff* asks whether M is diffeomorphic to S^n. But by the conjecture in *Top*, we can see that M is homeomorphic to S^n. So our question refines to the following:

Do there exist non-standard differential structures on S^n?

In general, these are called *exotic spheres*. In the case $n = 7$, there are in fact 28 different smooth structures. As we can see, it is still an open question whether non-standard differential structures on S^4, and in fact the *PL* case in $n = 4$ is equivalent to this question.

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[1]By a theorem of Whitehead, each smooth manifold admits a canonical PL structure.