EULER CHARACTERISTICS OF REAL ALGEBRAIC MANIFOLDS

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1. Introduction

The following question has motivated a large body of research in the 20th century. János Kollár refers to this problem as the *recognition problem* [Kol01]. Let X be a real algebraic variety (that is, the vanishing locus of some number of real polynomials in Euclidean space)

The recognition problem 1.1. What topological properties of $X(\mathbb{R})$ can be determined from the algebraic geometry of X? That is, can topological data about the topological space $X(\mathbb{R})$ be read off from the defining polynomials of X?

In general this should be an immensely difficult problem. An understanding of the homotopy type of $X(\mathbb{R})$ would encapsulate, for example, the homotopy type of varieties of the form $\left\{(x_1, \ldots, x_n) \colon \sum_i x_i^2 = 1\right\}$, that is, the homotopy groups of spheres. Understanding singular cohomology should be difficult in general as well. However this is a fascinating question to ask, and it would seem magical for any topological properties to be detectable via completely algebraic methods. Today we'll talk about a specific example, building off of work of Szafraniec and others in the 1990's, which demonstrates how to compute the *Euler characteristic* of a smooth real algebraic manifold in terms of its defining polynomials.

2. Warmup: root counting

Let $f(x) \in \mathbb{R}[x]$ be a polynomial of degree n. Let $X = V(f)$ be the vanishing locus of f, and consider the following topological spaces:

$$X(\mathbb{C}) := \{z \in \mathbb{C} \colon f(z) = 0\}$$
$$X(\mathbb{R}) := \{x \in \mathbb{R} \colon f(x) = 0\}.$$

Q: What is $\chi(X(\mathbb{C}))$?

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A: It depends on whether f has repeated roots! The Euler characteristic won't pick up multiplicity. So we should throw out any repeated roots that we see. Algebrao-geometrically, this is saying that $\chi(X) = \chi(X_{\text{red}})$ for any topological variety.

Working under the assumption that f has no repeated roots, we have that $\chi(X(\mathbb{C})) = n$, by the fundamental theorem of algebra.

Q: What is $\chi(X(\mathbb{R}))$?

A: It is not obvious! We would have to factor f, see how many degree one and degree two irreducible factors there are. This is not terribly difficult, since *univariate* polynomials can be factored in polynomial complexity (it's not quite as hard as factoring integers). However before the age of computers this was a daunting task.

In 1853, Hermite came up with a way to approach this problem. He developed a *(non-degenerate) symmetric bilinear form* over $\mathbb{R}$, whose *signature* recovered the number of real roots of f. Let's explain what these terms mean.

Any symmetric bilinear form $\beta : V \times V \to \mathbb{R}$, where V is a finite-dimensional vector space, has the property that V can be given a basis so that the Gram matrix of β in that basis is a diagonal matrix, with $+1$'s and -1's along the diagonal. This is called *Sylvester's law of intertia*. The *signature* of this form is the sum over the diagonal elements of such a form. It is a classical fact that isomorphism classes of symmetric bilinear forms are classified over $\mathbb{R}$ by their rank and signature.

What is this form? We build the algebra $Q := \mathbb{R}[x]/f(x)$, which is finite-dimensional over $\mathbb{R}$. Given any element $g \in Q$, multiplication by g induces an $\mathbb{R}$-linear map $Q \xrightarrow{m_g} Q$. Taking the trace of this map gives us an element in $\mathbb{R}$, so we have an $\mathbb{R}$-linear form:

$$\begin{aligned} Q &\to \mathbb{R} \\ g &\mapsto \operatorname{Tr}(m_g). \end{aligned}$$

This gives us a symmetric bilinear form, which we might call the *Hermite bilinear form*

$$\begin{aligned} \operatorname{Her}(f) : Q \times Q &\to \mathbb{R} \\ (g, h) &\mapsto \operatorname{Tr}(m_{g\cdot h}). \end{aligned}$$

Theorem 2.1. (Hermite) The signature of $\operatorname{Her}(f)$ is the number of real roots of f (that is, the Euler characteristic of $X(\mathbb{R})$).

Example 2.2. Consider $f(x) = (x^2 + 1)(x - 3) = x^3 - 3x^2 + x - 3$. Then $Q = \mathbb{R}[x]/f(x)$ has an $\mathbb{R}$-basis given by $\{1, x, x^2\}$. So the Gram matrix of $\operatorname{Her}(f)$ is the matrix whose (i, j)th entry is given by taking the trace of multiplication by $\beta_i\beta_j$. Explicitly,

$\operatorname{Her}(f) =$

	1	x	x^2
1	$\operatorname{Tr}(\mathrm{m}_{1\cdot 1})$	$\operatorname{Tr}(\mathrm{m}_{1\cdot x})$	$\operatorname{Tr}(m_{1\cdot x^2})$
x	$\operatorname{Tr}(m_{x\cdot 1})$	$\operatorname{Tr}(m_{x\cdot x})$	$\operatorname{Tr}(m_{x\cdot x^2})$
x^2	$\operatorname{Tr}(m_{x^2\cdot 1})$	$\operatorname{Tr}(m_{x^2\cdot x})$	$\operatorname{Tr}(m_{x^2\cdot x^2})$.

Multiplication by 1 is the identity, so it has trace 1. Multiplication by x is of the form

$$m_x = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix},$$

which has trace 3. This is going to get annoying to compute traces of multiplication by higher elements, so maybe we can use a trick? However we can remember that the trace of a matrix is a sum of its eigenvalues! The eigenvalues here are $3, i, -i$. So we see that

$$\begin{aligned} \operatorname{Tr}(m_{x^j}) &= \operatorname{Tr}\left(m_x^j\right) = 3^j + i^j + (-i)^j \\ &= 3^j + \begin{cases} 0 & j \text{ odd} \\ 2 & j \equiv 0 \pmod 4 \\ -2 & j \equiv 2 \pmod 4 \end{cases} \end{aligned}$$

So our Hermite form is

$$\operatorname{Her}(f) = \begin{array}{c|ccc} & 1 & \text{x} & \text{x}^2 \\ \hline 1 & 1 & 3 & 7 \\ \text{x} & 3 & 7 & 27 \\ \text{x}^2 & 7 & 27 & 83 \end{array}.$$

Diagonalizing this form, we obtain the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This has signature equal to 1, which is the number of real roots.

Takeaway: What did we have to do in this?

- Compute an $\mathbb{R}$-basis for Q
- Express $m_x : Q \to Q$ in this basis and take its trace
- Find the eigenvalues of m_x, use these to get $\operatorname{Tr}(m_{x^j})$
- Diagonalize the form $\operatorname{Her}(f)$ and recover its signature.

No step here is computationally costly! This can all be done in polynomial time.

Complaint: But wait, you can factor a univariate polynomial in polynomial time also, so why would you *ever* use this method?

Q: Given $f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n]$ so that their common zero locus is a loose collection of points, how many real points lie in $V(f)$?

Big issue: There is no reasonable algorithm for locating all the common roots of $(f_1, \ldots, f_n)$ when $n \gg 2$. Nothing is that much better than throwing a dart at Euclidean space and trying to Newton's method wherever it lands.

Theorem 2.3. (Pedersen, Roy, Szpirglas, 1993, [PRS93]) The signature of the Hermite form

$$\mathrm{Her}(f) : \frac{\mathbb{R}[x_1, \ldots, x_n]}{(f_1, \ldots, f_n)} \times \frac{\mathbb{R}[x_1, \ldots, x_n]}{(f_1, \ldots, f_n)} \to \mathbb{R}$$

yields the number of real roots. In particular the Euler characteristic of $X(\mathbb{R})$, when $X = V(f_1, \ldots, f_n)$. This computation can be done in polynomial time.

Idea: If you only care about *counting the roots* (i.e. the Euler characteristic) and you don't much care what they are, then symmetric bilinear form methods are a good way to move forward.

Big rough philosophy: Topological things over the reals that are communicated through algebraic data should *often* be signatures of symmetric bilinear forms.

3. Euler characteristics of smooth algebraic manifolds

The following section is based off [Sza89]. Let $F_1, \ldots, F_k \in \mathbb{R}[x_1, \ldots, x_n]$ define a smooth algebraic manifold in $\mathbb{R}^n$ (that is, the rank of the differential matrix DF is k at every point in $V(F)$). Let $\omega : \mathbb{R}^n \to \mathbb{R}$ be given by

$$\omega(x_1, \ldots, x_n) = \frac{1}{2}\left(x_1^2 + \ldots + x_n^2\right).$$

Then we can understand the manifold $W := V(F)$ by using ω as a Morse function. In particular, the critical values of ω along W provide us an understanding fo the topology of W. By Lagrange multipliers, this happens exactly when

$$x_j = \frac{\partial \omega}{\partial x_j} = \sum_{i=1}^{k} (-\lambda_i) \frac{\partial F_i}{\partial x_j},$$

for each $1 \leq j \leq n$. Recall that at such a critical point, we could take a coordinate system nearby, and look at the sign of the Hessian of ω. This will be $(-1)^s$, where s is the *Morse index* of ω at this point. In particular the Euler characteristic can be given by

$$\chi(W) = \sum_{p \in \mathrm{Crit}(\omega)} (-1)^{\mathrm{ind}_p \omega}.$$

This sum requires us to solve for the critical points though, which is something we'd like to avoid doing if possible. The insight of Szafraniec is as follows — rather than solving for the Lagrange multipliers, we should treat them as variables in their own right!

So instead of $W \subseteq \mathbb{R}^n$, we are instead going to move over to a bigger space:

$$\mathbb{R}^{n+k} = \{(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k)\}.$$

Lemma 3.1. The restricted Morse function $\omega|_W$ will have a critical point at $(x_1, \ldots, x_n)$ if and only if there is a uniquely determined point $(\lambda_1, \ldots, \lambda_k)$ so that

$$\text{grad}\omega(x) + \sum_{i=1}^{k} \lambda_i \text{grad} F_i = 0.$$

So what points do we care about in $\mathbb{R}^{n+k}$? We care about the points $(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k)$ where

$$x_1 = \sum_{i=1}^{k} \lambda_i \frac{\partial F_i}{\partial x_1}$$

$$\vdots$$

$$x_n = \sum_{i=1}^{k} \lambda_i \frac{\partial F_i}{\partial x_n},$$

and we also want to be on W, so we want $F_1(x_1, \ldots, x_n), \ldots, F_k(x_1, \ldots, x_n)$ to be equal to zero. Another way of phrasing all this is that we are looking for zeros of the map

$$H : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$$

$$(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k) \mapsto \left(x + \sum_{i=1}^{k} \frac{\partial F_i}{\partial x}, F(x) \right).$$

What is shocking and not at all obvious is that *how* H vanishes at a point $(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_k)$ encodes the Morse index of ω at this point!

Lemma 3.2. Assume ω has a critical point at $x \in \mathbb{R}^n$, and let $\lambda \in \mathbb{R}^k$ be the uniquely determined Lagrange multipliers at this point. Then

$$\deg_{(x,\lambda)} H = \operatorname{sgn} \det [DH(x, \lambda)] = (-1)^{s+k},$$

where s is the Morse index at x.

Proof idea. Take local coordinates, do some calculus magic, remember Cramer's rule from lin alg. □

This lemma is the real technical heart of the paper. From this we get the main theorem.

Theorem 3.3. We have that

$$\chi(W) = (-1)^k \deg(H).$$

That is, the Euler characteristic of a smooth real algebraic manifold can be computed as a global Brouwer degree of a polynomial function.

Brouwer degrees over the reals are always signatures of symmetric bilinear forms. Szafraniec provides a brief discussion of how you might compute such forms, and recent work of myself, Stephen McKean and Sabrina Pauli [BMP21] provides some rapid code for such things. So let's work through some examples — let's see how the algebra can detect the topological difference between two planar curves.

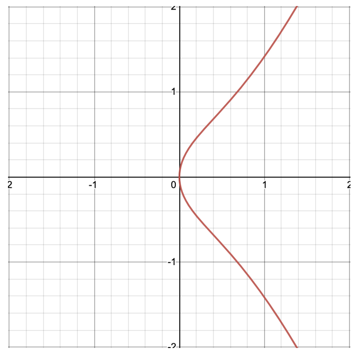

FIGURE 1. $y^2 = x(x^2+1)$

In this first example, we have $F(x, y) = y^2 - x^3 - x$. Adjoining a single Lagrange multiplier λ, we have that

$$\begin{aligned} H(x, y, \lambda) &= \left(x + \lambda \frac{\partial F}{\partial x}, y + \lambda \frac{\partial F}{\partial y}, F(x, y) \right) \\ &= \left(x + \lambda \left(-3x^2 - 1 \right), y + 2\lambda y, y^2 - x^3 - x \right). \end{aligned}$$

We see that $\operatorname{sgn} \deg(H) = -1$, so that $(-1)^k \operatorname{sgn} \deg(H) = 1 = \chi(W)$, since the curve is contractible.

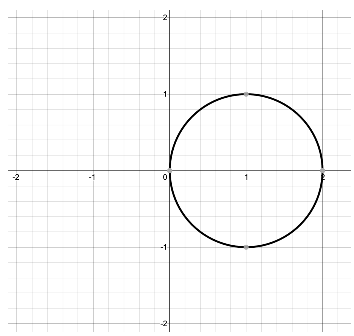

FIGURE 2. $y^2 + (x-1)^2 = 1$

For this second example, we have $F(x, y) = y^2 + x^2 - 2x$. This gives

$$H(x, y, \lambda) = \left(x + \lambda (2x - 2), y + \lambda 2y, y^2 + x^2 - 2x \right).$$

We can see that

$$\operatorname{sgn}\deg(H) = 0 = \chi(W).$$

While these two examples are easy to visualize, if you are given the vanishing of 17 functions in 90 variables, this strategy above gives you a feasible way to ascertain what the Euler characteristic of their vanishing locus is.

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