DWYER–KAN LOCALIZATION

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Abstract. Notes from an expository talk on Dwyer–Kan localization, given in the UPenn infinity-categories seminar in fall 2020.

References:

- nLab — simplicial localization
- Dongryul Kim : Dwyer–Kan localization

Goals/motivating questions:

- Why is ordinary localization $\mathcal{M}[W^{-1}]$ bad?
- How can we localize ∞-categories?
- Does localization translate across the homotopy coherent nerve adjunction?
- Morally, why does it make sense to look at the ∞-category underlying a model category?

1. Introduction

Let $\mathcal{M}$ be a model category. Then its homotopy category, denoted $\mathrm{Ho}(\mathcal{M})$, is defined to be $\mathcal{M}[W^{-1}]$. Beyond the problems we have previously discussed, there is another severe issue with the homotopy category.

Problem: We have that $\mathrm{Ho}(\mathcal{M})$ does not capture "higher order structure."

Quillen [Qui67] referred to the failure of $\mathrm{Ho}(\mathcal{M})$ to capture "higher order structure," by which we was referring to work of Spanier [Spa63] and others. Spanier's *Higher order operations* (1963) laid out the ideas of mapping cones and Puppe exact sequences which would functorially induce exact sequences. Spanier neglected to develop these in an abstract context, but instead remarked that they should hold for based spaces, quasi-topological spaces, and chain complexes.

We want a notion of localizing at weak equivalences that doesn't forget this structure.

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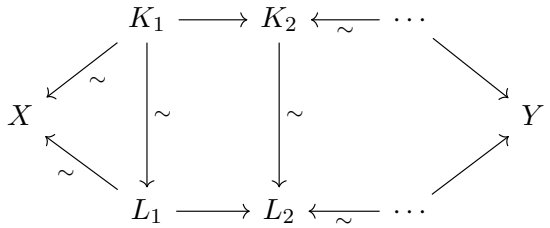
2. Hammock localization

Let $(\mathscr{C}, W)$ be a category with weak equivalences, not assumed to be small, and let $X, Y \in \mathscr{C}$ be any two objects. For any $n \geq 0$, define $H_n(X, Y)$ to be the category whose objects are length n zig-zags of morphisms in $\mathscr{C}$ of the form

$$X \xleftarrow{\sim} K_1 \to K_2 \xleftarrow{\sim} K_3 \to \cdots \to Y,$$

and whose morphisms are "hammocks" of the form



where each of the vertical maps are in W. Define a simplicial set by taking the coproduct over all the nerves of these categories, and quotient by an equivalence relation[1]

$$L^H\mathscr{C}(X, Y) := \amalg_{n\geq 0} N(H_n(X, Y))/\sim$$

Then there is a composition map

$$L^H\mathscr{C}(X, Y) \times L^H\mathscr{C}(Y, Z) \to L^H\mathscr{C}(X, Z),$$

given by concatenating hammocks.

Definition 2.1. The above construction gives a simplicially enriched category $L^H\mathscr{C}$, called the *hammock localization* of $(\mathscr{C}, W)$.

Proposition 2.2. [DK80a, p. 3.1] We have an equivalence of categories

$$\mathrm{Ho}\left(L^H\mathscr{C}\right) \simeq \mathscr{C}[W^{-1}].$$

So the hammock localization is a nice way to get a tractable description of the localization of a category. It is this description that we will bump up into the world of infinity-categories.

Proposition 2.3. [DK80b] The hammock localization $L^H\mathcal{M}$ of an arbitrary model category captures this "higher order information."

2.1. Hammock localization for simplicial categories.

Q: What if $\mathscr{C}$ and $\mathcal{W}$ are already simplicial categories?

If $\mathscr{C}, \mathcal{W} \in \mathtt{Cat}_\Delta$, and $\mathcal{W} \subseteq \mathscr{C}$ is a subcategory, then $L^H(\mathscr{C}, \mathcal{W})$ is the same as taking cofibrant replacements $\widetilde{\mathscr{C}} \rightarrowtail \mathscr{C}$ and $\widetilde{\mathcal{W}} \to \mathcal{W}$, and considering the naive localization $\widetilde{\mathscr{C}}[\widetilde{\mathcal{W}}^{-1}]$

[1] Inserting identities, removing them, composing them

levelwise. We provide another characterization of Dwyer–Kan localization in the setting of simplicial categories. We can define a naive localization as a functor

$$\begin{aligned} L : \mathtt{Cat}_\Delta \times \mathbb{1} &\to \mathtt{Cat}_\Delta \\ (\mathcal{W} \to \mathcal{C}) &\mapsto \mathcal{C}[\mathcal{W}^{-1}], \end{aligned}$$

where $\mathcal{C}[\mathcal{W}^{-1}] \in \mathtt{Cat}_\Delta$ is defined by inverting levelwise.

This is not completely correct, since it might fail to preserve weak equivalences. The correct version of this statement is that Dwyer–Kan localization is a *derived version* of this localization above.

Proposition 2.4. Dwyer–Kan localization in $\mathtt{Cat}_\Delta$ is obtained by first applying a cofibrant replacement, and then the localization L as above.

Explicitly, suppose we have a map $\mathcal{W} \to \mathscr{C}$ in $\mathtt{Cat}_\Delta$ we want to localize. Then if there is a diagram of the form

$$\begin{array}{ccc} \widetilde{\mathcal{W}} & \xrightarrow{p} & \mathcal{W} \\ {\scriptstyle \widetilde{i}}\downarrow & & \downarrow \\ \widetilde{\mathscr{C}} & \xrightarrow[q]{} & \mathscr{C}, \end{array}$$

where both p and q are cofibrant replacements, and $\widetilde{i}$ is a cofibration, then $L^H(\mathscr{C}, \mathscr{W}) \simeq \widetilde{\mathscr{C}}[\widetilde{\mathcal{W}}^{-1}]$.

3. ∞-LOCALIZATION

Recall that the inclusion $\mathtt{Kan} \hookrightarrow \mathtt{qCat}$ is a fully faithful embedding, and moreover, that it admits both left and right adjoints

$$\mathtt{Kan} \underset{K}{\overset{L}{\underset{\xrightarrow{i}}{\rightleftarrows}}} \mathtt{qCat}$$

Since L has a fully faithful right adjoint, it is a *localization* [Lur09, p. 5.2.7.2], while K is the functor assigning to a quasicategory its maximal Kan subcomplex. By formal nonsense we have an adjunction

$$\mathcal{L} := iL : \mathtt{qCat} \rightleftarrows \mathtt{qCat} : iK = \mathcal{K}.$$

We call $\mathcal{L}$ an ∞-*localization functor*, assigning to a quasicategory a Kan fibrant replacement. The idea is that it inverts edges in your quasi-category.

Question: Given a quasi-category, how should we define a class of weak equivalences that we'd like to invert?

Answer: We define them as a quasi-category $\mathscr{C}$, equipped with a set of arrows in $\mathscr{C}$, meaning a subset $W \subseteq \mathrm{Fun}(\Delta^1, \mathscr{C})$. This is called a *marked* ∞*-category.*

Denote by $\mathrm{Fun}^W(\mathscr{C}, \mathscr{D})$ the full subcategory of functors $\mathscr{C} \to \mathscr{D}$ sending edges in W to equivalences in $\mathscr{D}$.

Definition 3.1. Let $\mathscr{C} \in \mathtt{qCat}$ be a quasi-category and $W \subseteq \mathrm{Hom}_{\mathtt{sSet}}(\Delta^1, \mathscr{C})$ a set of edges. Then the ∞-*localization* of $\mathscr{C}$ with respect to W is a quasi-category $\mathscr{C}[W^{-1}]$, and a functor $f : \mathscr{C} \to \mathscr{C}[W^{-1}]$, satisfying a universal property that for any other $\mathscr{D} \in \mathtt{qCat}$, we have that

$$f^* : \mathrm{Fun}(\mathscr{C}[W^{-1}], \mathscr{D}) \xrightarrow{\sim} \mathrm{Fun}^W(\mathscr{C}, \mathscr{D})$$

is an equivalence.

Definition 3.2. We say a marked pair $(\mathscr{C}, W)$ is *saturated* if there is a map $f : \mathscr{C} \to \mathscr{D}$ of ∞-categories so that W is precisely the preimage of the equivalences in $\mathscr{D}$.

Suppose we have a marked pair $(\mathscr{C}, W)$, defined by a map $f : W \to \mathscr{C}$ and another quasi-category X. We denote the localization of $(\mathscr{C}, W)$ as $\mathcal{L}(f)$. Then the functors $\mathscr{C} \to X$ for which W is sent to equivalences should factor through $\mathcal{L}(f)$. Phrased differently, if we have $\mathscr{C} \to X$ so that $W \hookrightarrow \mathscr{C} \to X$ factors through the maximal Kan subcomplex $\mathcal{K}(X)$, then such a map should factor through the localization.

Consider the functor in X defined by the fiber product

$$\mathrm{Map}(\mathscr{C}, X) \times_{\mathrm{Map}(W,X)} \mathrm{Map}(W, \mathcal{K}(X)).$$

By our discussion above, this functor sends X to the mapping space $\mathrm{Map}(\mathcal{L}(f), X)$. So a good candidate for what the localization should be is any quasicategory corepresenting the functor above.

Proposition 3.3. We have that

$$\mathcal{L}(f) := \mathcal{L}(W) \amalg_W \mathscr{C}$$

corepresents the functor $\mathrm{Map}(\mathscr{C}, -) \times_{\mathrm{Map}(W,-)} \mathrm{Map}(W, \mathcal{K}(-))$.

Proposition 3.4. The total localization $\mathcal{L}(W)$ is the fibrant replacement of (W, W_1) in the category $\mathtt{sSet}^+$ of marked simplicial sets.

Corollary 3.5. The localization $\mathcal{L}(f)$ of $f : W \to \mathscr{C}$ is the fibrant replacement of $(\mathscr{C}, W)$ in $\mathtt{sSet}^+$.

Proposition 3.6. For any quasi-category $\mathscr{C}$ and any collection of edges W, the localization $\mathscr{C}[W^{-1}]$ exists.

We should see that this is "the same" as Dwyer–Kan localization. Explicitly, given a model category $\mathcal{M}$, we can do two things

(1) take its nerve, take W to be the marking of edges on the quasi-category $N(M)$, and then take it ∞-localization $N(\mathcal{M})[W^{-1}]$

(2) take its hammock localization $L^H(\mathscr{M}, W)$, which is a simplicial category, then take its homotopy coherent nerve $\mathcal{N}$.

These should (ideally) yield the same thing.

4. Unifying Dwyer–Kan localization and ∞-localization

Definition 4.1. Let $\mathscr{D}$ be a category, and consider the Dwyer–Kan localization $\mathscr{D} \to L^H(\mathscr{D}, \mathscr{D})$ obtained by inverting everything. This is called the *total Dwyer–Kan localization.*

Recall we referred to *total localization of ∞-categories* as the functor $\mathcal{L} = iL : \mathtt{qCat} \to \mathtt{qCat}$, where L was left adjoint to the inclusion functor $i : \mathtt{Kan} \hookrightarrow \mathtt{qCat}$. We had that $\mathcal{L}$ was some Kan fibrant replacement for a quasi-category.

We first compare total localization in each of these settings.

Lemma 4.2. [DK80c, p. 9.1] Let $\mathscr{D}$ be a simplicial category. Then the total Dwyer–Kan localization

$$\mathscr{D} \to L^H(\mathscr{D}, \mathscr{D})$$

has the property that if you first apply a fibrant replacement $\mathtt{Cat}_\Delta \to \mathtt{Cat}_\Delta$, and then the homotopy coherent nerve, the resulting map of quasi-categories is a total localization of ∞-categories.

Proposition 4.3. We have that the left adjoint $\mathfrak{C}[-]$ to the homotopy coherent nerve preserves localization.

Proof sketch. Let $f : W \to \mathscr{C}$ be a map in $\mathtt{qCat}$ given by a saturated marking on $\mathscr{C}$, and let $\mathcal{L}(f)$ be its localization. Everything is cofibrant in the Joyal model structure on simplicial sets, so W and $\mathscr{C}$ are cofibrant, and we may see that f is a monomorphism, and hence a cofibration. Consider the pushout diagram in $\mathtt{qCat}$

$$\begin{array}{ccc} W & \xrightarrow{f} & \mathscr{C} \\ \downarrow & \ulcorner & \downarrow \\ \mathcal{L}(W) & \longrightarrow & \mathcal{L}(f). \end{array}$$

Apply $\mathfrak{C} : \mathtt{qCat} \to \mathtt{Cat}_\Delta$ to this diagram. Since $\mathfrak{C}$ is a left adjoint, it preserves colimits, so we have a pushout diagram of simplicial categories

$$\begin{array}{ccc} \mathfrak{C}(W) & \rightarrowtail & \mathfrak{C}(\mathscr{C}) \\ \downarrow & \ulcorner & \downarrow \\ \mathfrak{C}(\mathcal{L}(W)) & \longrightarrow & \mathfrak{C}(\mathcal{L}(f)). \end{array}$$

Moreover, $\mathfrak{C}$ is a Quillen adjunction, so it preserves cofibrations, so $\mathfrak{C}[W] \to \mathfrak{C}[\mathscr{C}]$ is a cofibration. By Lemma 4.2 we know that $\mathfrak{C}[W] \to \mathfrak{C}[\mathcal{L}(W)]$ is a total localization. Therefore we conclude that $\mathfrak{C}[\mathcal{L}(f)]$ is a Dwyer–Kan localization of $\mathfrak{C}[\mathscr{C}]$ with respect to $\mathfrak{C}[W]$. □

We want to see the homotopy coherent nerve $\mathcal{N}$ preserves localizations.

Recall: If $S \in \mathtt{qCat}$, then if we take

$$S \mapsto \mathfrak{C}[S] \xrightarrow{\text{fibrant replacement}} \mathscr{D} \mapsto \mathcal{N}(\mathscr{D}),$$

then there is an induced map $S \to \mathcal{N}(\mathscr{D})$ which is an equivalence.

Proposition 4.4. If $\mathscr{C}, \mathcal{W}$ are fibrant simplicial categories, and $\mathcal{W} \subseteq \mathscr{C}$ a subcategory with $\mathrm{ob}\mathcal{W} = \mathrm{ob}\mathscr{C}$, then we have that the localization of $\mathcal{N}(\mathscr{C})$ by $\mathcal{N}(\mathcal{W})$ is computed by

- taking the hammock localization $L^H(\mathscr{C}, \mathcal{W})$
- fibrantly replacing it
- taking its homotopy coherent nerve.

5. ∞-categories underlying model categories

Let $\mathcal{M}$ be an arbitrary model category, with weak equivalences W. By what we have said above, $\mathcal{M}[W^{-1}]$ is poorly behaved and doesn't contain higher structures we might want. As we have seen, the workaround for this is given by Dwyer–Kan in the 80's, which is that we define the hammock localization $L^H(\mathcal{M}, W)$ which is a simplicial category whose mapping spaces are now simplicial sets, and contain this extra structure. Under the Quillen equivalence

$$\mathfrak{C} : \mathtt{qCat} \rightleftarrows \mathtt{Cat}_\Delta : \mathcal{N},$$

we have that localization of ∞-categories on the left, and Dwyer–Kan localization on the right, are morally the same. Explicitly, if we take $L^H(\mathcal{M}, W)$ on the right, fibrantly replace it, then take its nerve, we get an ∞-category which contains all the structure we would want, and is an excellent place to study the model category.

Definition 5.1. For a model category $\mathcal{M}$ with weak equivalences W, we define the ∞-*category underlying* $\mathcal{M}$ to be $\mathbf{R}\mathcal{N}(L^H(\mathcal{M}, W))$.

How does this compare to other definitions we may have seen:

Proposition 5.2. [DK80b, pp. 4.7, 4.8] We have that

$$\mathcal{M}^\circ \simeq L^H\mathcal{M}$$

are connected by an equivalence of $(\infty, 1)$-categories.

If $\mathcal{M}$ is a simplicial model category, then as we have discussed, we have that $\mathcal{M}^\circ$, the subcategory of fibrant-cofibrant objects, is equivalent to the Dwyer–Kan localization $L^H(\mathcal{M}, W)$. Moreover, since $\mathcal{M}^\circ$ is already enriched over $\mathtt{Kan}$, it is already fibrant in the Bergner model

structure; meaning we don't have to fibrantly replace it. So *the ∞-category underlying any simplicial model category* is $\mathcal{N}(\mathcal{M}^\circ)$.

6. Combinatorial model categories and presentable ∞-categories

Proposition 6.1. [Lur17, p. 1.3.4.22] If $\mathcal{A}$ is a combinatorial model category, then the underlying ∞-category of $\mathcal{A}$ is presentable.

Proof. By Dugger's Theorem, every combinatorial model category is equivalent to a combinatorial simplicial model category. Then we use the fact that, for any combinatorial simplicial model category $\mathcal{A}$, we have $\mathcal{N}(\mathcal{A}^\circ)$ is presentable [Lur09, A.3.7.6] □

This has interesting applications for computing colimits.

Proposition 6.2. [Lur17, p. 1.3.4.24] Let $\mathcal{A}$ be a combinatorial model category, and I an indexing category. Let $f : I \to \mathcal{A}^c$ be a diagram valued in the full subcategory of cofibrant objects. Let $\alpha : \mathrm{colim}_i F(i) \to X$ be an arrow in $\mathcal{A}$. Then the following are equivalent

(1) α exhibits X as $\mathrm{hocolim}(f)$
(2) the associated cocone from $\mathcal{N}(J)^\triangleright$ to the underlying ∞-category of $\mathcal{A}$ is an ∞-colimit.

The same is true for limits. Remember that all presentable ∞-categories are the underlying ∞-category of some combinatorial (simplicial) model category. This gives us the following slogan.

Slogan 6.3. Homotopy (co)limits in combinatorial model categories are (co)limits in presentable ∞-categories.

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