DIAGONALIZING SYMMETRIC BILINEAR FORMS

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Abstract. In these notes we record some of the basic maneuvers for manipulating symmetric bilinear forms and quickly recognizing their isomorphism class in $\mathrm{GW}(k)$.

1. Introduction

In the world of $\mathbb{A}^1$-homotopy theory, symmetric bilinear forms occupy a beautiful space, arising as elements of the 0th stable stem, and thus as Brouwer degrees of maps of varieties, $\mathbb{A}^1$-Euler characteristics, etc. When working through computations, we often need to quickly recognize the class of a symmetric matrix in the Grothendieck–Witt ring, without resorting to such extreme means as typing a 7×7-matrix into Sage or Macaulay2 and diagonalizing it there.

Sparse matrices and certain matrices with a lot of symmetry can be understood rather painlessly if one knows what to look for. In these notes we record some basic properties enjoyed by Gram matrices defining classes in the Grothendieck–Witt ring $\mathrm{GW}(k)$, and some tips and tricks for quickly recognizing patterns in these matrices.

2. Row and column operations

We recall that *diagonalizing a symmetric bilinear form is not the same as diagonalizing a matrix*. While matrices are diagonalized using matrix similarity, we have that symmetric bilinear forms are diagonalized via *matrix congruence*. Congruence preserves symmetry, while similarity does not.

Proposition 2.1. By performing the same row operation and the same column operation on a Gram matrix, we do not change its class in $\mathrm{GW}(k)$.

Example 2.2. This tells us that we can perform row operations and then the same column operations in order to diagonalize forms. Consider for example the form

$$\begin{pmatrix} 2 & 3 \\ 3 & -1 \end{pmatrix}.$$

Performing the row operation $R_1 \to 3\cdot R_2 + R_1$ yields

$$\begin{pmatrix} 11 & 0 \\ 3 & -1 \end{pmatrix},$$

after which we must perform the identical column operation $C_1 \to 3\cdot C_2 + C_1$ to obtain

$$\begin{pmatrix} 11 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus we have successfully diagonalized.

3. Block anti-diagonal forms

Some forms we might encounter are zero near the top left and bottom right, leaving blocks along the main anti-diagonal. We claim that such forms admit nice reduction properties.

Proposition 3.1. Consider the block Gram matrix given by

$$\beta = \begin{pmatrix} 0 & 0 & A \\ 0 & B & 0 \\ A^T & 0 & 0 \end{pmatrix},$$

where A is any invertible $n \times n$ matrix, and B is a $k \times k$ symmetric block matrix. Then we have that

$$\beta \cong n\mathbb{H} + B$$

in $\mathrm{GW}(k)$ (where here we are conflating the notation of the symmetric matrix B with its associated isomorphism class in GW).

Proof. Let's name some basis elements:

	a_1	a_2	$\cdots$	a_n	b_1	b_2	$\cdots$	b_k	α_1	α_2	$\cdots$	α_n
a_1	0	0	$\cdots$	0	0	0	$\cdots$	0	a_{11}	a_{12}	$\cdots$	a_{1n}
a_2	0	0	$\cdots$	0	0	0	$\cdots$	0	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_n	0	0	$\cdots$	0	0	0	$\cdots$	0	a_{n1}	a_{n2}	$\cdots$	a_{nn}
b_1	0	0	$\cdots$	0	b_{11}	b_{12}	$\cdots$	b_{1k}	0	0	$\cdots$	0
b_2	0	0	$\cdots$	0	b_{21}	b_{22}	$\cdots$	b_{2k}	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
b_k	0	0	$\cdots$	0	b_{k1}	b_{k2}	$\cdots$	b_{kk}	0	0	$\cdots$	0
α_1	a_{11}	a_{21}	$\cdots$	a_{n1}	0	0	$\cdots$	0	0	0	$\cdots$	0
α_2	a_{12}	a_{22}	$\cdots$	a_{n2}	0	0	$\cdots$	0	0	0	$\cdots$	0
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
α_n	a_{1n}	a_{2n}	$\cdots$	a_{nn}	0	0	$\cdots$	0	0	0	$\cdots$	0

Then the form is given by

$$\beta = \sum_{1 \leq i,j \leq n} a_{ij}(a_i \alpha_j + \alpha_i a_j) + \sum_{1 \leq i,j \leq k} b_{ij} b_i b_j.$$

We can define $\psi_i = \sum_{j=1}^n a_{ij}\alpha_j$. Then the form can be rewritten as

$$\sum_{i=1}^n (\psi_i a_i + a_i \psi_i) + \sum_{1 \leq i,j \leq k} b_{ij} b_i b_j.$$

Then in the basis $\{a_1, \ldots, a_n, \psi_1, \ldots, \psi_n, b_1, \ldots, b_k\}$, we have that β is expressible as n copies of the hyperbolic element plus the class of B. To verify this is indeed a basis, we check that the change-of-basis matrix is given by

$$(\vec{a}\ \vec{\psi}\ \vec{b}) = \begin{pmatrix} I_n & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_n \end{pmatrix} \cdot \begin{pmatrix} \vec{a} \\ \vec{\alpha} \\ \vec{b} \end{pmatrix},$$

which has determinant $\det(A) \neq 0$. □

Example 3.2. Observe that Proposition 3.1 allows us to "pick off" elements along an antidiagonal, extracting symmetric matrices out of the center. Consider for example the following form:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 4 & 5 & 0 \\ 0 & 5 & 8 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

By the result above, this is the same as $\mathbb{H} + \begin{pmatrix} 4 & 5 \\ 5 & 8 \end{pmatrix}$, which is much easier to diagonalize.

Example 3.3. When there is "no block" in the center, we can rapidly recognize a form as being hyperbolic:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 7 & 8 & 9 \\ 1 & 4 & 7 & 0 & 0 & 0 \\ 2 & 5 & 8 & 0 & 0 & 0 \\ 3 & 6 & 9 & 0 & 0 & 0 \end{pmatrix}.$$

This is of the form $\begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix}$, and is therefore hyperbolic, i.e. it is isomorphic to $3\mathbb{H}$.

This subdividing into blocks works particularly well with block upper triangular forms, which appear very naturally as local residue forms of univariate functions.

4. Upper triangular forms

Definition 4.1. We say a Gram matrix defines an *upper triangular* form if all the entries below the main anti-diagonal vanish.

Example 4.2. The following form, with the main anti-diagonal labeled in blue, is upper triangular:

$$\beta = \begin{pmatrix} 6 & -1 & 5 & 0 & 3 \\ -1 & -7 & 2 & 7 & 0 \\ 5 & 2 & 1 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Shockingly, nothing above the main anti-diagonal matters for the isomorphism class of the form.

Proposition 4.3. Given an $n \times n$ symmetric bilinear form $\beta = (a_{ij})$ which is upper triangular, we have that

$$\beta \cong \begin{cases} \frac{n}{2}\mathbb{H} & n \text{ even} \\ \frac{n-1}{2}\mathbb{H} + \left\langle a_{\frac{n+1}{2},\frac{n+1}{2}} \right\rangle & n \text{ odd.} \end{cases}$$

Proof. A citable reference is [KW20, Lemma 6]. The authors there assume that the form β is *Hankel*, meaning that it is constant along the anti-diagonals. Their proof does not exploit this symmetry however, and it extends to the more general upper triangular setting, as we remark in [BM21, A.2, 4.3]. □

Example 4.4. The following form is hyperbolic, since it is upper triangular and even rank:

$$\begin{pmatrix} 5 & 7 & -1 & 3 \\ 7 & -2 & 6 & 0 \\ -1 & 6 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{pmatrix} \cong 2\mathbb{H}.$$

Example 4.5. We have that

$$\begin{pmatrix} 6 & -1 & 5 & 0 & 3 \\ -1 & -7 & 2 & 7 & 0 \\ 5 & 2 & 1 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \end{pmatrix} \cong 2\mathbb{H} + \langle 1 \rangle,$$

where the $\langle 1 \rangle$ is coming from the $(3,3)$ entry right in the middle of the matrix.

5. Block upper triangular forms

Given a bilinear form β, suppose that it can be decomposed into blocks $\beta = (A_{ij})$, so that β is upper triangular *as a block matrix*. For example consider the matrix:

$$\beta = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 2 & 3 & 4 \\ 1 & -1 & 5 & 6 & 0 & 0 \\ 2 & 0 & 6 & 4 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is not upper triangular, since every time we see a "4," it occurs below the main anti-diagonal. Considering it as a 3×3 block matrix, it is what we might call *block upper triangular*. It turns out that β can be rewritten as follows:

$$\beta \cong 2\mathbb{H} + \begin{pmatrix} 5 & 6 \\ 6 & 4 \end{pmatrix}.$$

That is, we can "pick out" the middle block, just like we did with the middle element in Proposition 4.3. This is true in general.

Remark 5.1. A block symmetric bilinear form $\beta = (A_{i,j})$ necessarily has the property that $A_{i,j} = A_{j,i}^T$, since β is a symmetric matrix. In particular, we see that $A_{i,i} = A_{i,i}^T$ for each i, so every block appearing on the main diagonal is itself a symmetric matrix.

Proposition 5.2. Let $\beta = (A_{i,j})$ be a *block upper triangular* matrix, with k blocks in each row and column, and so that each $A_{i,j}$ is an $n \times n$ matrix. Then we have that

$$\beta \cong \begin{cases} \frac{nk}{2}\mathbb{H} & k \text{ even} \\ \frac{n(k-1)}{2}\mathbb{H} + A_{\frac{k+1}{2},\frac{k+1}{2}} & k \text{ odd,} \end{cases}$$

where in this last case we are conflating $A_{\frac{k+1}{2},\frac{k+1}{2}}$ with the class that it represents in $\mathrm{GW}(k)$. We note that it will define a symmetric matrix by Remark 5.1.

Proof. A proof may be found in [BM21, Lemma 4.5], with intuition in [BM21, Appendix A]. The lemma there states this in the setting of some Hankel-type symmetries, however the proof works in the absence of these symmetries, as remarked in [BM21, A.2]. □

This leads us to the following very rough philosophy.

Slogan 5.3. One can treat blocks like elements when diagonalizing symmetric bilinear forms.

6. Code

If all else fails, the following snippet of code, extracted from [BMP], will diagonalize a symmetric matrix M over $\mathbb{Q}$ in Sage.

```
# Turn a diagonal matrix into a list of its diagonal entries
def diagonal_matrix_to_list(M):
    list_of_entries = []
    for i in range(0,M.nrows()):
        list_of_entries.append(M[i,i])
    return(list_of_entries)

# Find the squarefree part of an integer
def strip_squares_integer(n):
    factorization = list(factor(n))
    reduced_factorization = []
    for pair in factorization:
        newpower = pair[1] % 2
        reduced_factorization.append([pair[0],newpower])

    m = factor(n).unit()
    for pair in reduced_factorization:
        m = m*(pair[0]**pair[1])
    return(m)

# Find a rational number modulo squares
def strip_squares_rational(q):
    a = q.numerator()
    b = q.denominator()
    n = a*b
    return strip_squares_integer(n)

# Match elements with their negatives into hyperbolic forms
def hyp_list(list_of_diagonal_entries):
    how_many_hyperbolics = 0
    leftover_stuff = []
    while list_of_diagonal_entries:
        x = list_of_diagonal_entries[0]
        y = -x
        if y in list_of_diagonal_entries:
            how_many_hyperbolics = how_many_hyperbolics + 1

            # Remove y from list
            del list_of_diagonal_entries[list_of_diagonal_entries.index(y)]
        else:
            leftover_stuff.append(x)
        # Remove x from list
        del list_of_diagonal_entries[0]

    if how_many_hyperbolics > 0 and len(leftover_stuff) > 0:
```

```
        return('Rational reduction: ' +str(how_many_hyperbolics) + #
        'H + < ' + ', '.join(map(str,leftover_stuff)) + ' >')
    elif how_many_hyperbolics == 0 and len(leftover_stuff) > 0:
        return('Rational reduction: < ' + ', '.join(map(str,leftover_stuff)) + ' >')
    elif how_many_hyperbolics > 0 and len(leftover_stuff) == 0:
        return('Rational reduction: ' +str(how_many_hyperbolics) + 'H')
    else:
        return('0')

# Take a diagonal matrix over Q and output its hyperbolic parts
def reduce_matrix(M):
    list_of_entries = diagonal_matrix_to_list(M)
    reduced_list_of_entries = []
    for q in list_of_entries:
        w = strip_squares_rational(q)
        reduced_list_of_entries.append(w)
    return(hyp_list(reduced_list_of_entries))
```

References

[BM21] Thomas Brazelton and Stephen McKean, *Lifts, transfers, and degrees of univariate maps*, 2021.

[BMP] Thomas Brazelton, Stephen McKean, and Sabrina Pauli, *a1-degree.sage*.

[KW20] Jesse Leo Kass and Kirsten Wickelgren, *A classical proof that the algebraic homotopy class of a rational function is the residue pairing*, Linear Algebra Appl. **595** (2020), 157–181. MR 4073493

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