

COMPLEX ORIENTATIONS

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ABSTRACT. Notes on complex orientations for an expository talk given in the chromatic homotopy theory seminar at UPenn in spring 2021.

References:

- nLab (as always)
- Lurie's notes on chromatic homotopy theory
- Hopkins, COCTALOS
- Carrick's lecture notes
- Chua's lecture notes
- Peter May MO, intuition behind the Thom class in parametrized homotopy theory
- Dugger - K theory notes on K -theory
- Ricardo Pedrotti
- Robert Bruner, MO post about the Thom isomorphism

1. INTRO ON ORIENTATIONS

Recall that an *orientation* of an n -manifold M is a class $\rho \in H_n(M)$ so that, for any point $x \in M$, the induced map

$$H_n(M) \rightarrow H_n(M, M - x) \simeq H_n(D^n, S^{n-1}) \simeq H_0(*) \cong \mathbb{Z}.$$

sends ρ to a generator. We think about a generator of $H_n(D^n, S^{n-1})$ as a way to assign a normal vector to the disk D^n around each point x in a continuous way, so that the normal vectors don't flip suddenly. This aligns with our intuition of what an orientation should mean.

More generally, let $V \rightarrow M$ be a vector bundle. Then for each point $p \in M$, we want to orient a small disk sitting over p in V . Suppose we have a Riemannian metric, and let $D(V)$ denote the disk bundle:

$$D(V) := \{v \in V : |v| \leq 1\}.$$

Then we are asking for a normal vector out of every disk over every point in M in a continuous way. In an analogous way to what we did above, taking D^n/S^{n-1} , we define

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$S(V)$ to be the sphere bundle

$$S(V) := \{v \in V : |v| = 1\},$$

and we consider the quotient bundle, also called the *Thom space*:

$$\mathrm{Th}(V) := D(V)/S(V).$$

For every point $x \in M$, we have a disk and its neighborhood sitting in V over x , which gives an inclusion

$$i_x : (D^n, S^{n-1}) \rightarrow \mathrm{Th}(V).$$

If V is a rank n bundle over M , then an orientation of V will be a class $\mu \in H_n(\mathrm{Th}(V))$ so that its restriction along i_x is a generator of $H_n(D^n, S^{n-1}) \cong \mathbb{Z}$.

Remark 1.1. Remarks about Thom spaces:

- (1) The Thom space of the pullback is the pullback of the Thom space
- (2) If $\varepsilon^n \rightarrow X$ is the trivial rank n bundle, then $\mathrm{Th}(\varepsilon^n) = S^n \wedge X_+$ is the trivial sphere bundle, but where all the points at ∞ in each sphere are glued together.
- (3) We have that $\mathrm{Th}(\mathbb{R}^n \oplus V) \simeq S^n \wedge \mathrm{Th}(V) = \Sigma^n V$.

Recall that a *ring spectrum* is a monoid in the stable homotopy category, that is, it has a multiplication map

$$E \wedge E \rightarrow E$$

which is unital and associative up to homotopy. We say a *ring map* is a morphism of spectra $f : E \rightarrow F$ so that the diagram commutes

$$\begin{array}{ccc} E \wedge E & \longrightarrow & E \\ f \wedge f \downarrow & & \downarrow f \\ F \wedge F & \longrightarrow & F. \end{array}$$

We remark that \mathbb{S} is initial among ring spectra.

Notation 1.2. We remark that the unique map $\mathbb{S} \xrightarrow{!} E$ defines a class in

$$\mathrm{Hom}(\Sigma^\infty S^0, E) = E^0(*).$$

We denote this class by $1 \in E^0(*)$. By abuse of notation, we can also call $1 \in E^2(S^2)$ under the suspension isomorphisms.

2. COMPLEX ORIENTABLE RING SPECTRA

For any base X and any vector bundle V , there is a vector bundle morphism in the diagram

$$\begin{array}{ccc} V & \longrightarrow & V \oplus \varepsilon^0 \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times X. \end{array}$$

Applying $\mathrm{Th}(-)$ on the top, we obtain a map $\mathrm{Th}(V) \xrightarrow{\Delta} \mathrm{Th}(V) \wedge X_+$, which we call the *Thom diagonal*.

Definition 2.1. Analogous to above, let $V \rightarrow X$ be any rank n bundle, and E any ring spectrum. Then we say that V is *E -oriented* if there is a morphism

$$\mu_V : \mathrm{Th}(V) \rightarrow \Sigma^n E,$$

(that is, a class in $E^n(\mathrm{Th}(V))$) so that for any $x \in X$, the inclusion of its fiber $S^n \hookrightarrow \mathrm{Th}(V)$ has the property that μ_V pulls back to a generator of $\pi_* E$.

Given such a Thom class, the following composite is an E -module isomorphism

$$\begin{array}{ccc} E \wedge \mathrm{Th}(V) & \xrightarrow{\sim} & E \wedge \Sigma^n X_+ \\ E \wedge \Delta \downarrow & & \uparrow \mathrm{id} \wedge \mu \wedge \mathrm{id} \\ E \wedge \mathrm{Th}(V) \wedge X_+ & \xrightarrow{\mathrm{id} \wedge \mu \wedge \mathrm{id}} E \wedge \Sigma^n E \wedge X_+ \xrightarrow{\sim} & E \wedge E \wedge \Sigma^n X_+ \end{array}$$

We call this the *Thom isomorphism*. This agrees with the normal Thom isomorphism in two ways

- (1) Applying π_* we get an isomorphism $E_*(\mathrm{Th}(V)) \xrightarrow{\sim} E_*(\Sigma^n X_+)$.
- (2) Since everything is a morphism of E -modules, we can apply $\mathrm{Hom}_{\mathrm{Mod}_E}(-, E)$, and recall that $\mathrm{Hom}_{\mathrm{Mod}_E}(E \wedge Y, E) \simeq \mathrm{Hom}_{S_p}(Y, E)$ to get that $[\Sigma^n X_+, E] \xrightarrow{\sim} [\mathrm{Th}(V), E]$, so there is an isomorphism $E^{*-n}(X_+) \cong E^*(\mathrm{Th}(V))$.

Definition 2.2. We say that a ring spectrum E is *complex orientable* if for every complex vector bundle $V \rightarrow X$ of real rank n , there is a *Thom class* $\mu_V : \mathrm{Th}(V) \rightarrow \Sigma^n E$ satisfying some compatibility conditions:

- (1) For $f : Y \rightarrow X$ and $V \rightarrow X$, we have that $\mu_{f^*V} = f^* \mu_V$.
- (2) We have that $\mu_{V \oplus V'} = \mu_V \cdot \mu_{V'}$ (here we are using that E is a ring spectrum to have this multiplication).

Exercise 2.3. Let E be a complex oriented ring spectrum, and let $f : E \rightarrow F$ be a ring map. Then F is complex oriented.

Now let E be complex oriented, and let $u \in \tilde{E}^2(\mathbb{C}P^\infty)$ be a cohomology class which is mapped to 1 under the composite

$$\tilde{E}^2(\mathbb{C}P^\infty) \rightarrow \tilde{E}^2(\mathbb{C}P^1) = E^0(*).$$

Then for each n , there is a map

$$E^*[u] \rightarrow E^*(\mathbb{C}P^n),$$

sending u to the pullback of u under the natural inclusion $\mathbb{C}P^n \subseteq \mathbb{C}P^\infty$.

Proposition 2.4. (Hopkins p.3) In the function above, u^{n+1} is mapped to 0.

Proof. Since u is a reduced cohomology class, when it is restricted to a contractible subspace, it becomes zero. We remark that $\mathbb{C}P^n$ can be covered by $n+1$ contractible open sets under its standard charts, call them each U_i . We recall that the cup product in relative cohomology works like

$$E^*(X, A) \times E^*(X, B) \mapsto E^*(X, A \cup B).$$

Thus by writing $u \in E^*(\mathbb{C}P^n, U_i)$ for each i , we have that

$$u^{n+1} \in E^*(\mathbb{C}P^n, U_1 \cup \dots \cup U_{n+1}) = E^*(\mathbb{C}P^n, \mathbb{C}P^n) = 0.$$

□

Lemma 2.5. We have that the map

$$E^*[u]/u^{n+1} \rightarrow E^*(\mathbb{C}P^n)$$

is a ring isomorphism.

Proof idea. Induct on n using the Atiyah–Hirzebruch spectral sequence

$$E_2^{p,q} = H^p(\mathbb{C}P^n, E^q(*)) \Rightarrow E^{p+q}(\mathbb{C}P^n),$$

to get the group isomorphism, and remark that the AHSS is multiplicative to get the ring isomorphism. □

Corollary 2.6. We have that

$$E^*(\mathbb{C}P^\infty) \cong E^*[[u]].$$

Moreover as E is multiplicative, we can prove that

$$E^*\left(\prod_{i=1}^n \mathbb{C}P^\infty\right) \cong E^*[[u_1, \dots, u_n]].$$

Theorem 2.7. (Chua, 3.7) There is a natural bijection between

- (1) Complex orientations of E
- (2) Classes $u \in E^2(\mathbb{C}P^\infty)$ that restrict to an E^* -module generator of $E^2(\mathbb{C}P^1) \cong E^0$.

Proof idea. Given any complex orientation, the Thom class of the tautological bundle $\mathcal{O}(-1) \rightarrow \mathbb{C}P^\infty$ gives a map $\text{Th}(\mathcal{O}(-1)) \rightarrow E$, defining a class in $E^2(\text{Th}(\mathcal{O}(-1))) \simeq E^2(\mathbb{C}P^\infty)$ which maps to 1 by hypothesis. Conversely, suppose we have a Thom class $u \in E^2(\mathbb{C}P^\infty)$. We want to use this to generate Thom classes for each complex bundle. Since rank n complex bundles are classified by maps $X \rightarrow \text{BU}(n)$, it will suffice to generate Thom classes for the universal bundles over $\text{BU}(n)$ for all n . We will come back to this. □

Remark 2.8. A complex orientation is equivalently a commutative diagram of the form

$$\begin{array}{ccc} S^2 & \longrightarrow & \Omega^\infty E \\ \downarrow & \nearrow \text{dashed} & \\ \mathbb{C}P^\infty & & \end{array}$$

Proposition 2.9. Let R be any ring. Then the EM spectrum HR is complex orientable.

Proof. We have that $H^2(S^2, R) \cong R$ is non-empty. Moreover, since $S^2 \hookrightarrow \mathbb{C}P^\infty$ is the inclusion of a 2-skeleton, we have an induced isomorphism

$$H^2(S^2, R) \xrightarrow{\cong} H^2(\mathbb{C}P^\infty, R).$$

Therefore any choice of generator of $H^2(S^2, R)$ is a complex orientation. \square

Thus if we have a map $S^2 \rightarrow \Omega^\infty E$, we can check stage by stage whether it lifts along the inclusions $S^2 = \mathbb{C}P^1 \subseteq \mathbb{C}P^2 \subseteq \mathbb{C}P^3 \subseteq \dots \subseteq \mathbb{C}P^\infty$. The obstruction to a lift of the form

$$\begin{array}{ccc} \mathbb{C}P^n & \longrightarrow & \Omega^\infty E \\ \downarrow & \nearrow \text{dashed} & \\ \mathbb{C}P^{n+1} & & \end{array}$$

is determined by a class in $\pi_{2n+1}\Omega^\infty E = \pi_{2n+1}E$.

Corollary 2.10. If E is concentrated in even degrees, then any choice of map $S^2 \rightarrow \Omega^\infty E$ is a complex orientation (i.e. it is always complex orientable).

Remark 2.11. We have that KU is complex orientable by Bott periodicity.

3. CHERN CLASSES AND FGLS

We recall that a complex line bundle over a space X is classified by a homotopy class of maps $X \rightarrow \mathbb{C}P^\infty$.

Definition 3.1. If E is a cohomology theory, complex oriented by a class $u \in E^2(\mathbb{C}P^\infty)$, then for any line bundle $L \rightarrow X$, defined by a map

$$f : X \rightarrow \mathbb{C}P^\infty$$

we define $c_1(L) \in E^2(X)$ by $c_1(L) := f^*(u)$.

In particular, the tautological line bundle over $\mathbb{C}P^\infty$ gives a map $\mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$. Taking the tensor product of this bundle with itself defines an element of

$$[\mathbb{C}P^\infty, \mathbb{C}P^\infty] \times [\mathbb{C}P^\infty, \mathbb{C}P^\infty] \cong [\mathbb{C}P^\infty \times \mathbb{C}P^\infty, \mathbb{C}P^\infty].$$

We denote by g this resulting map. Then g^* induces a map on cohomology

$$g^* : E^*(\mathbb{C}P^\infty) \rightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty).$$

If E is complex orientable, this is a map of the form

$$E^*[[u]] \rightarrow E^*[[x, y]].$$

Thus the image of u is some formal power series, and actually is a formal group law.

Remark 3.2. Every complex orientation of a multiplicative cohomology theory defines a formal group law in $E^*[[x, y]]$.

Just as we can tensor line bundles, we can add them. The map $L_1, \dots, L_n \mapsto \bigoplus_{i=1}^n L_i$ is classified by a map

$$\theta : \mathrm{BU}(1) \times \dots \times \mathrm{BU}(1) \rightarrow \mathrm{BU}(n),$$

which is homotopy equivariant. Applying $[-, H\mathbb{Z}]$ we get a map

$$H^*(\mathrm{BU}(n); \mathbb{Z}) \rightarrow H^*(\mathrm{BU}(1)^n; \mathbb{Z}) \cong (H\mathbb{Z})^*[[t_1, \dots, t_n]] \cong \mathbb{Z}[t_1, \dots, t_n],$$

which factors through the subspace $\mathbb{Z}[t_1, \dots, t_n]^{\Sigma_n} \subseteq \mathbb{Z}[t_1, \dots, t_n]$ consisting of the equivariant polynomials under permutation of the t_i 's. Let c_i denote the i th symmetric polynomial on the t_i 's:

$$\begin{aligned} c_1 &= t_1 + \dots + t_n \\ c_2 &= \sum_{i \neq j} t_i t_j \\ &\vdots \\ c_n &= t_1 \cdots t_n. \end{aligned}$$

Then our construction above defines a function (which is really an isomorphism)

$$\theta^* : H^*(\mathrm{BU}(n); \mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}[c_1, \dots, c_n].$$

By abuse of notation, let $c_i := (\theta^*)^{-1}(c_i)$. Now let X be any space, and consider a rank n vector bundle on it given by $X \rightarrow \mathrm{BU}(n)$. Then c_i maps to a class in $H^*(X; \mathbb{Z})$ under restriction. This is the i th *Chern class*.

Remark 3.3. An analogous discussion says that for any complex oriented cohomology theory there is an isomorphism

$$E^*(\mathrm{BU}(n)) \cong E^*[[c_1, \dots, c_n]].$$

Again, for any $X \rightarrow \mathrm{BU}(n)$ we get Chern classes in $E^*(X)$.

We can now wrap up the proof of the theorem above.

Since $E^*(\mathrm{BU}(n-1)) \cong E^*[[c_1, \dots, c_{n-1}]]$, and $E^*(\mathrm{BU}(n)) \cong E^*[[c_1, \dots, c_n]]$, under the natural inclusion $\mathrm{BU}(n-1) \hookrightarrow \mathrm{BU}(n)$, we can view c_n as a relative cohomology class

$$c_n \in E^*(\mathrm{BU}(n), \mathrm{BU}(n-1)).$$

Let $\zeta : EU(n) \rightarrow \mathrm{BU}(n)$ denote the universal bundle. This comes with an action by $U(n)$ whose quotient is $\mathrm{BU}(n)$. Since the unit disk bundle contracts to a zero section, we have that $\mathrm{BU}(n) \simeq D(\zeta)$. Since $U(n-1) \subseteq U(n)$, it also induces an action on $EU(n)$. This gives a fiber sequence

$$\begin{array}{ccccc} U(n)/U(n-1) & \hookrightarrow & EU(n)/U(n-1) & \twoheadrightarrow & EU(n)/U(n) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ S^{2n-1} & \hookrightarrow & \mathrm{BU}(n-1) & \twoheadrightarrow & \mathrm{BU}(n). \end{array}$$

Therefore $\mathrm{BU}(n-1) \rightarrow \mathrm{BU}(n)$ is a sphere bundle, and is actually isomorphic to the unit sphere bundle $S(\zeta)$ sitting inside of $EU(n) \rightarrow \mathrm{BU}(n)$. Thus we have that

$$c_n \in E^*(\mathrm{BU}(n), \mathrm{BU}(n-1)) \simeq E^*(D(\zeta), S(\zeta)) \simeq E^*(\mathrm{Th}(\zeta)).$$

Proposition 3.4. We have that c_n defines a Thom class for the universal bundle over $\mathrm{BU}(n)$.

Thus from our complex orientation u we can define c_n , which gives us a Thom class for the universal rank n bundle, which will give Thom classes for all rank n bundles.

4. REPRESENTABILITY OF COMPLEX ORIENTATIONS

We just used the Thom space of the universal bundle over $\mathrm{BU}(n)$ to construct universal Thom classes. This construction turns out to be more general, providing another characterization of complex orientations.

Definition 4.1. We define $\mathrm{MU}(n) := \mathrm{Th}(EU(n) \rightarrow \mathrm{BU}(n))$. By what we have seen above, we have that $\mathrm{MU}(n) \simeq \mathrm{BU}(n)/\mathrm{BU}(n-1)$.

In particular since the n th Chern class was an element $c_n \in E^n(\mathrm{BU}(n), \mathrm{BU}(n-1))$, we can rephrase the n th Chern class as a map

$$\Sigma^\infty \mathrm{MU}(n) \xrightarrow{c_n} E.$$

As n varies, we can form $\mathrm{MU}(n)$ into a spectrum, though there is an indexing issue. The structure maps come from the observation that

$$\Sigma^2 \mathrm{MU}(n) = \Sigma^2 \mathrm{Th}(EU(n)) = \mathrm{Th}(\mathbb{C} \oplus EU(n)) = \mathrm{Th}(EU(n+1)) = \mathrm{MU}(n+1).$$

So we define a spectrum MU by

$$\begin{aligned}\mathrm{MU}_{2n} &:= \mathrm{MU}(n) \\ \mathrm{MU}_{2n+1} &:= \Sigma \mathrm{MU}(n).\end{aligned}$$

Remark 4.2. We have that

- (1) $\mathrm{MU}(0) \simeq \mathbb{S}$
- (2) $\mathrm{MU}(1) \simeq \Sigma^{-2}\Sigma^\infty\mathbb{C}P^\infty$.

We also have that MU is multiplicative. Since there is a map

$$\mathrm{BU}(n) \times \mathrm{BU}(m) \rightarrow \mathrm{BU}(n+m)$$

which classifies direct sums, this induces a map $\mathrm{MU}(n) \wedge \mathrm{MU}(m) \rightarrow \mathrm{MU}(n+m)$. To bootstrap this up to a multiplication on spectra, we define

$$\begin{aligned}(\mathrm{MU} \wedge \mathrm{MU})_{2n} &= \mathrm{MU}_n \wedge \mathrm{MU}_n \\ (\mathrm{MU} \wedge \mathrm{MU})_{2n+1} &= \mathrm{MU}_{n+1} \wedge \mathrm{MU}_n,\end{aligned}$$

and check that these multiplications define a morphism $\mu : \mathrm{MU} \wedge \mathrm{MU} \rightarrow \mathrm{MU}$ (see Carrick 2.4). This is homotopy associative and unital, and moreover is homotopy commutative, so MU is an E_∞ -ring spectrum.

Proposition 4.3. We have that MU is complex oriented.

Proof. For any complex bundle $V \rightarrow X$ of complex rank n , it is classified by some map f so that

$$\begin{array}{ccc} V & \longrightarrow & \mathrm{EU}(n) \\ \downarrow & \lrcorner & \downarrow \zeta_n \\ X & \xrightarrow{f} & \mathrm{BU}(n). \end{array}$$

Applying Th , this gives us a map $\mathrm{Th}(\zeta_n) \rightarrow \mathrm{Th}(V)$, that is, $\mathrm{MU}(n) \rightarrow \mathrm{Th}(V)$. We may check that pulling back c_n along this map gives a Thom class in $H^{2n}(\mathrm{Th}(V))$. We see that such Thom classes are functorial along pullbacks, induce Thom isomorphisms, and are multiplicative (since MU is a ring spectrum). \square

If E is complex oriented, we said that every Thom class was pulled back from some Chern class

$$c_n \in E^*(\mathrm{BU}(n), \mathrm{BU}(n-1)) \simeq E^*(D(\zeta_n), S(\zeta_n)) \simeq E^*(\mathrm{MU}(n)).$$

That is, the Chern class for E is given by some map $\phi_n : \mathrm{MU}(n) \rightarrow \Sigma^{2n}E$. Since $\mathrm{MU}(n)$ is concentrated in degree 0 here, this is the same as asking for a space map $\mathrm{MU}(n) \rightarrow E_{2n}$. We can ask to what extent the ϕ_n 's agree levelwise.

Proposition 4.4. We have that the diagram commutes up to homotopy

$$\begin{array}{ccc} \Sigma^2 \text{MU}(n) & \xrightarrow{\Sigma^2 \phi_n} & \Sigma^2 E_{2n} \\ \downarrow & & \downarrow \\ \text{MU}(n+1) & \xrightarrow{\phi_{n+1}} & E_{2n+2}. \end{array}$$

Proof. We can induct on n . Beginning with $n = 0$, we have a map

$$\phi_0 : \text{MU}(0) \simeq S^0 \rightarrow E_0,$$

representing $c_0 := 1$. Suppose that we have maps ϕ_k for $k \leq n$ yielding the Chern classes, which commute in the sense above. By definition (or by base case), c_1 commutes with the classes 1, that is, we have a diagram

$$\begin{array}{ccc} & S^2 & \\ & \swarrow & \searrow \\ \text{MU}(1) & \xrightarrow{\phi_1} & E_2. \end{array}$$

This is because $\text{MU}(1)$ is just $\mathbb{C}P^\infty$. Smashing ϕ_1 with ϕ_n , we get a homotopy commutative diagram

$$\begin{array}{ccc} S^2 \wedge \text{MU}(n) & \xrightarrow{\phi_n} & S^2 \wedge E_n \\ 1 \times \text{id} \downarrow & & \downarrow 1 \times \text{id} \\ \text{MU}(1) \wedge \text{MU}(n) & \xrightarrow{\phi_1 \wedge \phi_n} & E_2 \wedge E_{2n}. \end{array}$$

Since $c_1 \cdot c_n = c_{n+1}$ (as universal bundles), we have that the bottom square on the diagram commutes up to homotopy

$$\begin{array}{ccc} S^2 \wedge \text{MU}(n) & \xrightarrow{\phi_n} & S^2 \wedge E_n \\ 1 \times \text{id} \downarrow & & \downarrow 1 \times \text{id} \\ \text{MU}(1) \wedge \text{MU}(n) & \xrightarrow{\phi_1 \wedge \phi_n} & E_2 \wedge E_{2n} \\ \downarrow & & \downarrow \\ \text{MU}(n+1) & \xrightarrow{\phi_{n+1}} & E_{2n+2}. \end{array}$$

□

Thus the Chern classes correspond to a general map of spectra

$$c : \text{MU} \rightarrow E.$$

Proposition 4.5. The map described above is a unital ring spectra homomorphism.

Proof. For unitality, we are asking that

$$\begin{array}{ccc} \mathbb{S} & \longrightarrow & \text{MU} \\ & \searrow & \downarrow c \\ & & E \end{array}$$

commutes up to homotopy. Levelwise, we have asking for the composite $S^{2n} \rightarrow \text{MU}(n) \xrightarrow{c_n} E_{2n}$ to be equivalent to 1, meaning that c_n restricts to a unit of E_{2n} on the fiber, which is precisely the statement that c_n is a Thom class.

For multiplicativity, we want to see that the diagram commutes

$$\begin{array}{ccc} \text{MU} \wedge \text{MU} & \xrightarrow{c \wedge c} & E \wedge E \\ \downarrow & & \downarrow \\ \text{MU} & \xrightarrow{c} & E. \end{array}$$

Up to some magic about smash products of spectra, this is the same as asking for diagrams of the form

$$\begin{array}{ccc} \text{MU}(n) \wedge \text{MU}(m) & \xrightarrow{c_n \wedge c_m} & E_{2n} \wedge E_{2m} \\ \downarrow & & \downarrow \\ \text{MU}(n+m) & \xrightarrow{c_{n+m}} & E_{2(n+m)}. \end{array}$$

But this follows from the Whitney product formula. □

Theorem 4.6. The following are in natural bijection

- (1) Ring maps $\text{MU} \rightarrow E$.
- (2) Complex orientations on E

Proof. We remark that the inclusion

$$\Sigma^{-2}\mathbb{C}P^\infty \simeq \text{MU}(1) \rightarrow \text{MU}$$

gives a class $t \in \text{MU}^2(\mathbb{C}P^\infty)$ which is a complex orientation. Moreover this is a universal one. We therefore have a map

$$\begin{aligned} \text{Hom}_{\text{Ring}}(\text{MU}, E) &\rightarrow \{\text{complex orientations on } E\} \\ \phi &\mapsto \phi(t). \end{aligned}$$

We can check that this is inverse to the construction above. □

Thus MU is the “universal complex oriented cohomology theory,” and its distinguished class $t \in \text{MU}^2(\mathbb{C}P^\infty)$ is the “universal complex orientation.”

Recall that for any complex oriented spectrum E , there was a map $g : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \rightarrow \mathbb{C}P^\infty$ which classified tensoring line bundles by the Yoneda lemma. If $u \in E^2(\mathbb{C}P^\infty)$ was the complex orientation, then we had an induced map

$$g^* : E^*[[u]] \simeq E^*(\mathbb{C}P^\infty) \rightarrow E^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) \simeq E^*[[x, y]].$$

The image $g^*(u)$ was some formal group law in $E^*[[x, y]]$, providing a connection between complex orientations and formal group laws.

Now let $t \in \text{MU}^2(\mathbb{C}P^\infty)$ be the universal group law. Then g induces a map

$$g^* : \text{MU}^*[[t]] \rightarrow \text{MU}^*[[x, y]],$$

sending t to some formal group law in x and y over the ring $\text{MU}^* = \pi_*\text{MU}$. We are curious what formal group law this is.

In order to understand what this is, we invoke Lazard's Theorem, and we see that the universal group law is represented by some particular ring map

$$L \rightarrow \pi_*\text{MU}.$$

Theorem 4.7. (Milnor—Quillen) The map above is a ring isomorphism.

We will spend the next few lectures (lectures 7—10 of Lurie) proving this statement. This result completely tells us what the homotopy of MU looks like. Moreover, this aligns with something we already knew: if E is complex orientable, it produces some formal group law. The complex orientation on E comes from some ring map $\text{MU} \rightarrow E$, inducing a ring homomorphism $\pi_*\text{MU} \rightarrow \pi_*E$. After proving this theorem, this is a map $L \rightarrow E^*$, which determines the formal group law associated to the complex orientation on E . The goal for this direction will be as follows:

- (1) (Lecture 7) Compute $H_*(\text{MU}; \mathbb{Z})$ first.
- (2) (Lecture 8) Determine a method for passing from integral homology to homotopy. This is given by the *Adams spectral sequence*.
- (3) (Lecture 9) Apply the Adams spectral sequence for MU to try to understand the E_2 -page
- (4) (Lecture 10) Use this to prove the Milnor—Quillen theorem.

We will study MU in a bit more detail then take a crack at the first step of this. Before doing so, we want to advertize some ideas that are coming up.

4.1. What is to come. Roughly speaking, there was a map

$$\{\text{complex orientations}\} \rightarrow \{\text{FGLs}\}.$$

We can ask to what extent this is surjective. That is, given a ring R and an FGL in $R[[x, y]]$, does it arise from a complex orientation on some ring spectrum E ? That is, does there exist a ring spectrum E so that $\pi_*(E) = R$, and a ring spectrum map $f : \text{MU} \rightarrow E$ so that $\pi_*(f) : L \rightarrow R$ classifies our given FGL? By Brown representability, this is equivalent to asking about the existence of a generalized homology theory satisfying certain axioms. This

problem is understood by *Landweber exact functor theorem* which provides necessary and sufficient conditions to understand the spectra that arise in this fashion, and an algebraic interpretation of the associated FGLs.

Another direction of study comes from the Adams—Novikov spectral sequence. Under certain conditions, for a spectrum X , we will have a spectral sequence converging to $\pi_{p-q}(X)$. The terms on the E_2 -page are group cohomology groups, with coefficients in $\mathrm{MU}_*(X)$, the MU-homology of X :

$$E_2^{*,q} \text{ is group cohomology with coefficients in the } L\text{-module } \mathrm{MU}_*(X).$$

Passing into the world of algebraic geometry, we can view $\mathrm{MU}_*(X)$ as a quasi-coherent sheaf over the moduli stack $\mathcal{M}_{\mathrm{fg}} = \mathrm{Spec} L/G$, where G is a group scheme which we could think of as acting via reparametrizing formal group laws. The E_2 -pages then turn into

$$E_2^{*,q} \text{ is } \textit{stack} \text{ cohomology of } \mathcal{M}_{\mathrm{fg}} \text{ with coefficients in the quasi-coherent sheaf } \mathrm{MU}_*(X).$$

This passage from L to $\mathcal{M}_{\mathrm{fg}}$ carries some information in how it relates to other objects. Let A be any ring:

We have that:	...correspond to:
Ring homomorphisms $L \rightarrow A$	FGLs over A
Scheme morphisms $\mathrm{Spec} A \rightarrow \mathrm{Spec} L$	FGLs over A
Stack morphisms $\mathrm{Spec} A \rightarrow \mathcal{M}_{\mathrm{fg}}$	“equivalence classes” of FGLs over A .

Understanding $\mathcal{M}_{\mathrm{fg}}$ is equivalent (by Yoneda) to understanding all maps into $\mathcal{M}_{\mathrm{fg}}$, which is equivalent to understanding formal group laws. So we will be really interested in the structure of $\mathcal{M}_{\mathrm{fg}}$ as a stack. We observe that $\mathcal{M}_{\mathrm{fg}}$ is not an algebraic stack since it doesn't admit a presentation by a locally of finite type scheme (since there are too many generators on L). It is, however, a filtered colimit of algebraic stacks

$$\mathcal{M}_{\mathrm{fg}} = \mathrm{colim}_n \mathcal{M}_{\mathrm{fg}}^n.$$

We could very roughly envision $\mathcal{M}_{\mathrm{fg}}^n$ as some stacky analogue of the n -skeleton of a CW structure on $\mathcal{M}_{\mathrm{fg}}^n$. The A -points of $\mathcal{M}_{\mathrm{fg}}^n$ correspond to formal group laws of “height” n . So studying this stratification by algebraic stacks corresponds to studying formal group laws by their height.