# COLIMITS IN QUASI-CATEGORIES

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Abstract. Notes from an expository talk given in the infinity-categories seminar at UPenn in fall 2020.

## 1. Introduction

1.1. **Goal.** We want to show how to compute colimits in quasi-categories using homotopy colimits in simplicial categories.

1.2. **The homotopy coherent nerve, again.** Recall that we defined a functor

$$\begin{aligned} \mathfrak{C} : \Delta &\to \mathtt{Cat}_\Delta \\ \Delta^n &\to \mathfrak{C}[\Delta^n], \end{aligned}$$

where $\mathfrak{C}[\Delta^n]$ was the simplicial category with

$$\begin{aligned} \mathrm{ob}\mathfrak{C}[\Delta^n] &= \mathrm{ob}[n] \\ \mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i,j) &= \begin{cases} \varnothing & j < i \\ N(P_{i,j}) & i \leq j. \end{cases} \end{aligned}$$

The *simplicial nerve* of a simplicial category $\mathscr{C}$ was characterized by

$$\mathrm{Hom}_{\mathtt{sSet}}(\Delta^n, N(\mathscr{C})) \cong \mathrm{Hom}_{\mathtt{Cat}_\Delta}(\mathfrak{C}[\Delta^n], \mathscr{C}).$$

This functor extended to a functor

$$\begin{aligned} \mathfrak{C} : \mathtt{sSet} &\to \mathtt{Cat}_\Delta \\ S &\mapsto \mathfrak{C}[S], \end{aligned}$$

and fit into an adjunction with the simplicial nerve

$$\mathfrak{C} : \mathtt{sSet} \rightleftarrows \mathtt{Cat}_\Delta : N.$$

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1.3. **Mapping spaces in quasi-categories.** In quasi-categories, and in simplicial sets more generally, we want more than just a set of morphisms between two objects, we want a space (e.g. a Kan complex) or more generally speaking, a homotopy type.

Given a simplicial set $S$, we define its *homotopy category* to be the homotopy category $h\mathfrak{C}[S]$ of the simplicial category $\mathfrak{C}[S]$ [Lur09, p. 1.1.5.14]. This is enriched over $\mathcal{H}$, the homotopy category of spaces. In particular, for any $x, y \in S$, we have that $\mathrm{Map}_{hS}(x, y)$ is the homotopy type of the mapping space $\mathrm{Map}_{\mathfrak{C}[S]}(x, y)$.

In general, we define the *mapping space* for any simplicial set $S$ and objects $x, y \in S$ to be

$$\mathrm{Map}_S(x, y) := \mathrm{Map}_{hS}(x, y).$$

The problem is that for arbitrary simplicial sets, this is not a Kan complex (this problem is rectified by defining "right morphisms" — the study of relating possible definitions of mapping spaces comes after the topic of *(un)straightening*). However, if $S = \mathscr{C}$ is a quasi-category, then it will be [Lur09, §2.2].

In general, given any fibrant simplicial category $\mathcal{D}$, its fibrancy means that it is enriched over $\texttt{Kan}$, so $\mathrm{Map}_{\mathscr{D}}(x, y) \in \texttt{Kan}$ for all $x, y \in \mathcal{D}$. A general strategy for getting a $\texttt{Kan}$ complex as a mapping space in an arbitrary simplicial set is

(1) take $x, y \in S \in \texttt{sSet}$
(2) take $S \mapsto \mathfrak{C}[S]$, and then fibrantly replace $\mathfrak{C}[S]$ by some fibrant simplicial category $\mathscr{D}$
(3) take the nerve of $\mathscr{D}$, so the entire process produces $S \to N(\mathscr{D})$.

This map produced is a Joyal equivalence, so we can kind of think of $\mathrm{Map}_S(x, y)$ as $\mathrm{Map}_{N(\mathscr{D})}(x, y)$.

1.4. **Colimits in 1-category theory.** Let $p : \mathcal{J} \to \mathscr{C}$ be a functor of 1-categories. Then recall that a *colimit* of $p$ is the nadir of an initial cone under the diagram. Explicitly, we can append an extra object $*$ to $\mathcal{J}$, and a morphism $j \to *$ for every $j \in \mathcal{J}$ to make a new category, which we denote by

$$J \star [0] = J^{\triangleright}.$$

Functors out of $J^{\triangleright}$ are exactly cones under $J$-shaped diagrams.

What should the "category of cones under $p$" look like? Clearly it should be somehow related to $\mathrm{Fun}(J^{\triangleright}, \mathscr{C})$, but this category yields all possible cones under all possible $J$-shaped diagrams in $\mathscr{C}$, and we only want cones under $p$.

We can see there is an inclusion $J \to J^{\triangleright}$, so the category of *cones under* $p$ is exactly the functors fitting into a commutative diagram

$$\begin{array}{ccc} J & & \\ \big\downarrow & \searrow^{p} & \\ J^{\triangleright} & \dashrightarrow & \mathscr{C}. \end{array}$$

This motivates the following definition.

**Definition 1.1.** The category of *cones under* $p$ is the functor category

$$\mathrm{Fun}_{J/\mathtt{Cat}}\left(J^{\triangleright}, \mathscr{C}\right),$$

where $J^{\triangleright}, \mathscr{C} \in \mathrm{Fun}(J/\mathtt{Cat})$ are understood as $(J \hookrightarrow J^{\triangleright})$ and $J \xrightarrow{p} \mathscr{C}$, respectively.

Then we say that a functor $\overline{p} : J^{\triangleright} \to \mathscr{C}$ is a *colimit cone* if it is initial in $\mathrm{Fun}_{J/\mathtt{Cat}}\left(J^{\triangleright}, \mathscr{C}\right)$. If such a functor exists, then we have $\overline{p} : J \star [0] \to \mathscr{C}$, and we have that $\overline{p}(0) =: \mathrm{colim} p$ is an object in $\mathscr{C}$, called the *colimit* of $p$.

There is another roundabout way to describe this category, which relates more to the notation used in HTT. Leave $p : J \to \mathscr{C}$ alone, but now instead of taking a join with a point, allow the flexibility to take a join with any category. In particular, for any $Y$, there is a canonical inclusion $p : J \to J \star Y$, so you can ask about the functor

$$\mathrm{Fun}_{J/\mathtt{Cat}}(J \star -, \mathscr{C}) : \mathtt{Cat}^{\mathrm{op}} \to \mathtt{Set}.$$

This functor turns out to be representable! That is, there is a category $\mathscr{C}_{p/}$, which the property that there is an isomorphism

$$\mathrm{Fun}_{\mathtt{Cat}}(Y, \mathscr{C}_{p/}) \cong \mathrm{Fun}_{J/\mathtt{Cat}}(J \star Y, \mathscr{C}),$$

which is natural in $Y$. Returning our attention to the case $Y = [0]$, we can see that $\mathscr{C}_{p/}$ is exactly equal to $\mathrm{Fun}_{J/\mathtt{Cat}}(J^{\triangleright}, \mathscr{C})$ — that is, it is the category of cones under $p$.

1.5. **Colimits in quasi-categories.** Let's take this whole discussion and bump it up to the world of quasi-categories.

There is a "join" of simplicial sets which can be thought of as follows — given $X, Y \in \mathtt{sSet}$, we have that $X \star Y$ is a "cone under $X$ with vertex $Y$." This is compatible with the join of categories in the following sense.

**Exercise 1.2.** Given categories $A, B \in \mathtt{Cat}$, we have that $N(A \star B) \cong N(A) \star N(B)$ is an isomorphism of simplicial sets.

We also have "slice" categories — given the fact that the nerve is fully faithful, we can consider it as inducing a functor $J/\mathtt{Cat} \to NJ/\mathtt{sSet}$ for any $J \in \mathtt{Cat}$.

In particular, let's consider the situation where $p : J \to \mathscr{C}$ is a functor from a simplicial set $J$ into a quasi-category $\mathscr{C}$. Then we claim that the functor

$$\mathrm{Hom}_{J/\mathtt{sSet}}(J \star -, \mathscr{C}) : \mathtt{sSet}^{\mathrm{op}} \to \mathtt{Set}$$

is representable, by a simplicial set $\mathscr{C}_{p/}$.

**Proposition 1.3.** If $\mathscr{C}$ is a quasi-category, then so is $\mathscr{C}_{p/}$.

Analogous to above, a *colimit cone* of $p$ should be an initial object in $\mathscr{C}_{p/}$. But remember that $\mathscr{C}_{p/}$ is now a quasi-category, so asking for an initial object is a more involved procedure.

**Proposition 1.4.** An object $a \in A$ in a locally small 1-category is *initial* if we have that $\mathrm{Hom}_A(a, a')$ is terminal in $\mathtt{Set}$ for all $a' \in A$.

Let's bump this up to quasi-categories.

**Proposition 1.5.** An object $d \in \mathscr{D}$ in a quasi-category is *initial* if and only if any of the equivalent conditions hold

- it is initial in $h\mathscr{D}$ (this is how Lurie defines initial, see [Lur09, p. 1.2.12.1])
- as $h\mathscr{D}$ is enriched over $\mathcal{H}$, this is equivalent to asking that $\mathrm{Map}_{h\mathscr{D}}(d, d')$ is terminal in $\mathcal{H}$ for every $d' \in \mathscr{D}$
- $\mathrm{Map}_{h\mathscr{D}}(d, d') \simeq *$ is contractible for all $d' \in \mathscr{D}$.

Thus an element morphism $\overline{p} \in \mathscr{C}_{p/}$ is *initial* if and only if

$$\mathrm{Hom}_{\mathscr{C}/p}(\overline{p}, q) \simeq *$$

for any other $q \in \mathscr{C}_{p/}$. By the universal property of $\mathscr{C}_{p/}$, we see that

$$\mathscr{C}_{p/} \simeq \mathrm{Hom}_{\mathtt{sSet}}([0], \mathscr{C}_{p/}) \simeq \mathrm{Hom}_{J/\mathtt{sSet}}(J^{\triangleright}, \mathscr{C}).$$

Thus we can view $\overline{p} \in \mathrm{Hom}_{J/\mathtt{sSet}}(J^{\triangleright}, \mathscr{C})$. The statement that $\overline{p}$ is initial in this category is the same as the statement that, for any other cone $q$ under $p$, the mapping space of natural transformations $[\overline{p}, q]$ is weakly contractible.

Somehow we want to relate this space of cones with a space of maps between the nadirs of the cones. This is accomplished by the following lemma.

**Lemma 1.6** (4.2.4.3)**.** Let $\mathscr{C}$ be a qcat, $K$ a simplicial set, $p : K \to \mathscr{C}$ a diagram, and $\overline{p} : K^{\triangleright} \to \mathscr{C}$ a cone under the diagram. Then the following are equivalent

(1) $\overline{p}$ is a colimit cone
(2) if $X \in \mathscr{C}$ is the image of the cone point under $\overline{p}$, then $\overline{p}$ determines a natural transformation $\alpha : p \Rightarrow \mathrm{const}_X$, where $\mathrm{const}_X : K \to \mathscr{C}$ is the constant diagram at $X$. Then for every $Y \in \mathscr{C}$, there is a homotopy equivalence

$$\phi_Y : \mathrm{Map}_{\mathscr{C}}(X, Y) \to \mathrm{Map}_{\mathrm{Fun}(K,\mathscr{C})}(p, \mathrm{const}_Y),$$

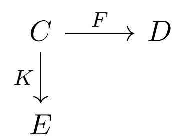
given by sending $X \to Y$ to a diagram $p \xrightarrow{\alpha} \mathrm{const}_X \to \mathrm{const}_Y$.

This is exactly the analog of the "universal property of the colimit" in quasi-categories.

## 2. Homotopy Colimits in Simplicial Cats

2.1. **Kan extensions.** (Riehl: Categorical Homotopy Theory)

Let $C, D, E$ be categories, and consider a diagram of the form




A general *lifting problem* we might be interested in is as follows: does there exist a functor $E \to D$ making the diagram commute?

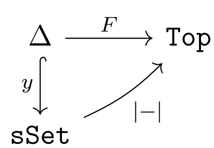
In general no, however we can ask a slightly different question. Is there a functor $E \to D$ which commutes up to natural isomorphism, and is initial/terminal among all such pairs of functors and natural isomorphisms? This is called a *Kan extension*.

**Definition 2.1.** A *left Kan extension* of $F$ along $K$ is a functor $\mathrm{Lan}_K F : D \to E$ with a natural transformation $\eta : F \Rightarrow \mathrm{Lan}_K F \circ K$ such that, for any other pair $(G, \gamma)$ we have that $\gamma$ factors uniquely through $\eta$.

Dually a right Kan extension is a functor $\mathrm{Ran}_K F : D \to E$ with the 2-cell going in the opposite direction. In general Kan extensions may or may not exist.

**Examples 2.2.**

(1) If $F : \Delta \to \mathtt{Top}$ is the functor sending $[n]$ to the topological $n$-simplex $\Delta^n$, then the left Kan extension along the Yoneda embedding exists, and is called *geometric realization.*




Moreover we can replace $\mathtt{Top}$ with any cocomplete category $\mathcal{E}$, and some categorical trickery shows that geometric realization is always a left adjoint to some functor $\mathcal{E} \to \mathtt{sSet}$.

(2) If $K$ is fixed, and $\mathrm{Lan}_K F$ and $\mathrm{Ran}_K F$ exist for any $F$, then they assemble into functors

$$\mathrm{Lan}_K(-), \mathrm{Ran}_K(-) : \mathrm{Fun}(C, E) \to \mathrm{Fun}(D, E).$$

Let $K^* : \mathrm{Fun}(D, E) \to \mathrm{Fun}(C, E)$ denote precomposition with $K$. Then there are adjunctions

$$\mathrm{Lan}_K(-) \dashv K^* \dashv \mathrm{Ran}_K(-).$$

This generates tons of examples.

(3) Let $f : R \to S$ be a ring homomorphism. Viewing it as a functor between one-object abelian categories, then the left Kan extension along a functor $R \to \mathrm{End}_{\texttt{Ab}}(M)$ is extension of scalars, which is left adjoint to restriction of scalars.

(4) Let $\Delta_{\leq n}$ be the subcategory of $\Delta$ with objects $\{[0], \dots, [n]\}$. Then the inclusion $i_n : \Delta_{\leq n} \hookrightarrow \Delta$ induces a functor $i_n^* : \texttt{sSet} \to \mathrm{Fun}(\Delta_{\leq n}^{\mathrm{op}}, \texttt{Set})$, called *$n$-truncation.* By the example above, this admits left and right adjoints, called the $n$-skeleton and $n$-coskeleton (abuse of notation).

(5) The inclusion of a subgroup $H \subseteq G$ induces restriction of representations $\texttt{Vect}^G \to \texttt{Vect}^H$. This admits left and right adjoints, called induction and coinduction.

2.2. **Derived functors.** Let $F : C \to D$ be a functor between model categories. Consider the following diagram

$$\begin{array}{ccc} C & \xrightarrow{F} & D \\ {\scriptstyle Q_C}\downarrow & & \downarrow{\scriptstyle Q_D} \\ hC & \dashrightarrow & hD. \end{array}$$

The definition of a *derived functor* in the model category setting is a suitable morphism $hC \to hD$, as in the diagram above. It is often too strict to ask for such a morphism to exist making the diagram strictly commute, but we can ask for the diagram to commute up to some 2-cell.

**Definition 2.3.** The *left derived functor* $\mathbb{L}F$ is the right Kan extension

$$\mathbb{L}F := \mathrm{Lan}_{Q_C}(Q_D \circ F).$$

The *right derived functor* $\mathbb{R}F$ is the left Kan extension

$$\mathbb{R}F := \mathrm{Lan}_{Q_C}(Q_D \circ F).$$

These may or may not exist. The following theorem provides sufficient conditions for the existence of functors. Recall a *Quillen adjunction* between model categories is an adjunction $F : C \rightleftarrows D : G$ such that $F$ preserves cofibrations and trivial cofibrations, or, equivalently, $G$ preserves fibrations and trivial fibrations.

**Theorem 2.4.** If $F : C \rightleftarrows D : G$ is a Quillen adjunction, then the derived functors $\mathbb{L}F$ and $\mathbb{R}G$ exist and form an adjoint pair [Rie, p. 4.2].

**Example 2.5.** Let $J$ be a small indexing diagram, which we will use to index diagrams of spaces. There is a constant functor

$$\texttt{Top} \xrightarrow{\Delta} \mathrm{Fun}(J, \texttt{Top}),$$

given by sending a space $X$ to the constant diagram at $X$ (all objects $j \in J$ are sent to $X$, and all morphisms in $J$ are sent to $\mathrm{id}_X$). Since $\texttt{Top}$ is complete and cocomplete, this functor admits left and right adjoints, which are the colimit and limit functors, respectively

$$\mathrm{colim} \dashv \Delta \dashv \lim .$$

We claim that, if they exist, the homotopy colimit and homotopy limit can be *defined* to be

$$\text{hocolim} := \mathbb{L}\text{colim}$$
$$\text{holim} := \mathbb{R}\lim .$$

But first we need to see they exist.

**Q**: Are colim $\dashv \Delta$ and $\Delta \dashv \lim$ Quillen adjunctions?

**A**: Yes — but you need to be careful which model structure you pick on $\text{Fun}(J, Top)$. For colim $\dashv \Delta$, you want the projective model structure, and for $\Delta \dashv \lim$, you want the injective model structure.

2.3. **Homotopy Kan extensions.** In order to understand colimits in quasi-categories, we would like to related them to homotopy colimits in simplicial categories. In order to do this, we need to understand what a "homotopy colimit" means in a simplicial category. Via the discussion above, we can understand homotopy colimits as Kan extensions along colimits.

The way to understand homotopy colimits in simplicial categories is by viewing the theory of Kan extensions as "`Set`-enriched Kan extensions" and then asking what `sSet`-enriched Kan extensions should be.

Let $f : C \to C'$ be a functor of small simplicially enriched categories, and let $\mathbf{A}$ be a combinatorial simplicial model category. Then the enriched functor category $\text{Fun}(C, \mathbf{A})$ can be endowed with a projective or an injective model structure. Let

$$f^* : \text{Fun}(C', \mathbf{A}) \to \text{Fun}(C, \mathbf{A})$$

denote precomposition with $f$. Then $f^*$ admits both right and left adjoints

$$f_! \dashv f^* \dashv f_*.$$

As in the colimit/limit situation, each of these will be a Quillen adjunction if we pick the suitable injective or projective model structure.

**Definition 2.6.** The right derived functor

$$\mathbb{R}f_* : (\text{Fun}(C, \mathbf{A})_{\text{inj}})^\circ \to \left(\text{Fun}(C', \mathbf{A})_{\text{inj}}\right)^\circ$$

is called the *homotopy right Kan extension*.

The left derived functor

$$\mathbb{L}f_! : (\text{Fun}(C, A)_{\text{proj}})^\circ \to \left(\text{Fun}(C', A)_{\text{proj}}\right)^\circ$$

is called the *homotopy left Kan extension*.

If $C' = *$, then $\text{Fun}(C', A) = \text{Fun}(*, A) = A$, so $f^*$ is precisely precomposition with a collapse map $C \to *$, that is, it is a functor

$$f^* : \mathbf{A} \to \text{Fun}(C, \mathbf{A}).$$

It is easy to see this is the *constant functor*, that is, $\Delta$. In this case, we see that $f_! = \operatorname{colim}$ and $f_* = \lim$, so we have that the *homotopy colimit* functor is

$$\operatorname{hocolim} = \mathbb{L}f_! : \operatorname{Fun}(C, A)^\circ_{\text{proj}} \to A^\circ,$$

and the *homotopy limit* functor is

$$\operatorname{holim} = \mathbb{R}f_* : \operatorname{Fun}(C, A)^\circ_{\text{inj}} \to A^\circ.$$

*This is the definition of homotopy (co)limit in a simplicial model category.*

2.4. **Coherent and commutative diagrams.** Suppose you have a model category $\mathcal{M}$ and a diagram $p : K \to \mathcal{M}$. Then we can take a *homotopy colimit* of $p$ just like we do in spaces. Since a homotopy colimit is invariant under natural weak equivalence of the diagram, we should really think about $p$ as a diagram $p : K \to \operatorname{Ho}(\mathcal{M})$ valued in the homotopy category. By taking its homotopy colimit, we have some type of cone diagram that looks like

$$\overline{p} : K^{\triangleright} \to \operatorname{Ho}(\mathcal{M}).$$

If we tried to lift this back to $\mathcal{M}$, it might not make sense — that is, by picking representatives for all the morphisms in the image of $\overline{p}$, we would get a diagram in $\mathcal{M}$ which commutes *only up to coherent homotopy.* Phrased differently, homotopy colimits give *homotopy coherent diagrams.*

The question of when a homotopy coherent diagram is equivalent to a genuine commutative one is a subtle question. In the context of model categories, this is easy, just cofibrantly replace everything in sight and take an honest colimit. However in simplicial categories, this is a much harder question.

If $\mathbf{A}$ is a combinatorial simplicial model category, then every homotopy coherent diagram can be replaced (straightened) by a genuine commutative one.

Given an arbitrary simplicial category $\mathscr{C}$, and a diagram valued in it, how could we straighten it? There is a result that the functor category $\operatorname{Fun}(\mathscr{C}^{\text{op}}, \texttt{sSet})$ is a combinatorial simplicial model category,[1] and that there is an enriched Yoneda embedding which is fully faithful $y : \mathscr{C} \hookrightarrow \operatorname{Fun}(\mathscr{C}^{\text{op}}, \texttt{sSet})$. We can exploit this embedding, and the straightening in combinatorial simplicial model categories, to prove the upcoming theorem.

## 3. Comparison theorem

The goal in this section is to understand the following slogan:

"Homotopy colimits in a simplicial category are colimits in a quasicategory."

[1] And a fun fact is that all combinatorial simplicial model categories arise as localizations of categories of this form (see Dugger's theorem)

**Theorem 3.1.** [Lur09, p. 4.2.4.1] Let $\mathscr{C}$ and $\mathcal{J}$ be fibrant simplicial categories, and $F : \mathcal{J} \to \mathscr{C}$ a simplicial functor. Suppose we are given an object $C \in \mathscr{C}$ and a compatible family of maps $\{\eta_i : F(i) \to C\}_{i \in \mathcal{J}}$. The following conditions are equivalent:

(1) the maps $\eta_i$ exhibit $C$ as a homotopy colimit of $F$
(2) letting $f := N(F) : N(\mathcal{J}) \to N(\mathscr{C})$ and $\overline{f} : N(\mathcal{J})^{\triangleright} \to N(\mathscr{C})$ the extension of $f$ determined by $\{\eta_i\}$, then $f$ is a colimit diagram in $N(\mathscr{C})$.

*Proof idea.* (1) Embed $\mathscr{C} \hookrightarrow \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathtt{sSet})$ via the enriched Yoneda lemma.
(2) Straighten the diagram out in that context, so it is an honest commutative diagram, but restrict your attention to objects in $\mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathtt{sSet})$ which are equal (or at least isomorphic to) representable objects.
(3) This colimit has a universal property described in terms of mapping spaces on these simplicial presheaf categories. By some categorical trickery, we can relate these mapping spaces (up to weak equivalence) to the mapping spaces on the genuine categories themselves.
(4) Since the mapping spaces in the simplicial categories $\mathcal{J}$ and $\mathscr{C}$ are, by definition, the mapping spaces of the associated quasi-categories $N(\mathcal{J})$ and $N(\mathscr{C})$, this gives a universal property up to weak equivalence on the mapping spaces in quasi-categories. This characterizes the colimit.

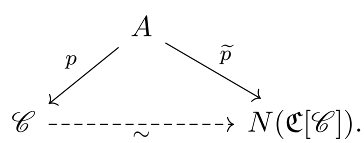
□

## 4. Computing limits and colimits

This follows [Lur09, §4.4.1].

**Corollary 4.1.** Let $\mathscr{D} \in \mathtt{Cat}_\Delta$ be any simplicial category, and let $p : J \to \mathscr{D}$ be a diagram which admits a colimit. Then $\mathrm{colim}(p) = \mathrm{colim} N(p) : N(J) \to N(\mathscr{D})$.

In general, let $A$ be a simplicial set, and $\mathscr{C}$ a quasi-category. Then to take a colimit of $p : A \to \mathscr{C}$, we need to first view $\mathscr{C}$ as the nerve of a simplicial category. Since $\mathscr{C} \xrightarrow{\sim} N(\mathfrak{C}[\mathscr{C}])$, we have a diagram




Then it suffices to understand the colimit of $\widetilde{p}$. In particular, we may always assume a quasi-category is the homotopy coherent nerve of a simplicial category.

### 4.1. Coproducts.

Let $\mathscr{C} \in \mathtt{qCat}$ be a quasi-category which is the nerve of a simplicial category $\mathscr{D}$, and let $A$ be an indexing set that we want to take a coproduct over. Then we can view $A$ as a discrete category, and take its nerve so that it lies in $\mathtt{sSet}$. We could also view it as lying in $\mathtt{Cat}_\Delta$ by being trivially enriched over $[0]$ or $\varnothing$.

Then any morphism $p : A \to \mathscr{D}$ picks out a collection of objects $\{X_a\}_{a \in A}$ in $\mathscr{D}$. An associated cocone $A^{\triangleright} \to \mathscr{D}$ picks out a point $X \in \mathscr{D}$, along with morphisms $X_\alpha \to X$. This will be a homotopy colimit if and only if, for every $Y \in \mathscr{C}$ we have that

$$\mathrm{Map}_{\mathscr{D}}(X, Y) \to \prod_{a \in A} \mathrm{Map}_{\mathscr{D}}(X_\alpha, Y)$$

is a weak equivalence. Thus, the associated morphism $A^{\triangleright} \to N(\mathscr{D}) \simeq \mathscr{C}$ picking out the points $X_\alpha$ and $X$ will be a colimit if and only if

$$\mathrm{Map}_{\mathscr{C}}(X, Y) \to \prod_{\alpha \in A} \mathrm{Map}_{\mathscr{C}}(X_\alpha, Y)$$

is a weak equivalence for every $Y \in \mathscr{C}$.

We denote by $\amalg_{\alpha \in A} X_\alpha$ such an $X$ above, and called it the *coproduct* in the quasi-category $\mathscr{C}$. We remark this is only defined up to weak equivalence.

**Example 4.2.** (Pushouts) A *square* in a quasi-category is a map of the form

$$(\Lambda^2_0)^{\triangleright} = \Delta^1 \times \Delta^1 \to \mathscr{C}.$$

A commutative diagram of the form

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

in a quasi-category $\mathscr{C}$ is then a pushout square if and only if it is a homotopy pushout square in $\mathfrak{C}[\mathscr{C}]$. That is, if and only if the associated diagram

$$\begin{array}{ccc} \mathrm{Map}_{\mathscr{C}}(Y, Z) & \longrightarrow & \mathrm{Map}_{\mathscr{C}}(Y', Z) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathscr{C}}(X, Z) & \longrightarrow & \mathrm{Map}_{\mathscr{C}}(X', Z) \end{array}$$

is a homotopy pullback square in `Kan` for every $Z \in \mathscr{C}$.

**Takeaway**: A colimit in a quasi-category is characterized by a universal property on mapping spaces in `Kan` up to weak equivalence.

## References

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