BUNDLES

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ABSTRACT. Personal notes on vector bundles typed in preparation for an oral exam in algebraic topology.

1. Fiber bundles

Theorem 1.1. Let $\pi : E \to B$ be a fiber bundle. If B is paracompact, then π is a Hurewicz fibration.

2. Vector bundles

Recall that a vector bundle is a fiber bundle for which the fibers have the structure of a real or complex vector space, compatible with local trivializations. This definition encourages us to consider the study of vector bundles as lying somewhere in the intersection of linear algebra and topology, and Hatcher has stated that we could call this theory Linear Algebraic Topology [Hat03].

Example 2.1. *(Clutching functions)* If we want to cook up vector bundles over a sphere S^k, one way that we could do this is to take trivial bundles on the (closed) northern and southern hemispheres and then find a way to glue them together along the equator which retains the vector bundle structure but twists it in some way.

We should glue each point on the equator to the other in a linear way, so for each point x lying on the equator S^{k-1}, we need a function f which assigns to x the transformation it undergoes passing from north to south. Explicitly, assuming that we are working with complex vector bundles, this means that $f(x) \in \mathrm{GL}_n(\mathbb{C})$. Moreover as we fluctuate x along the equator, we should not undergo severe changes in the assigned matrices, hence we should want f to be continuous. Finally it is easy to check that a homotopy of f does not change the isomorphism class of the bundle, thus we get a map

$$[S^{k-1}, \mathrm{GL}_n(\mathbb{C})] \to \mathtt{Vect}^n_{\mathbb{C}}(S^k).$$

This map is in fact an isomorphism. This is great, and it looks similar to things we recognize from algebraic topology, like the representability of certain functors. The problem is that the left side is not a functor of S^k.

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Recall that, for a topological group G we have that $\Omega BG \simeq G$, so let's rewrite the above isomorphism as follows:

$$\begin{aligned}[S^{k-1}, \mathrm{GL}_n(\mathbb{C})] &= [S^{k-1}, \Omega\mathrm{BGL}_n(\mathbb{C})] \\ &= [\Sigma S^{k-1}, \mathrm{BGL}_n(\mathbb{C})] \\ &= [S^k, \mathrm{BGL}_n(\mathbb{C})].\end{aligned}$$

Thus the set of n-dimensional complex vector bundles on any sphere is represented by homotopy classes of maps into $\mathrm{BGL}_n(\mathbb{C})$.

In the real case, we note that $\mathrm{GL}_n(\mathbb{R})$ is not path-connected, and if $k > 1$, then the equator S^{k-1} is path-connected, and its image via a clutching function will lie in one of the path components of $\mathrm{GL}_n(\mathbb{R})$ (which are determined by the sign of the determinant). To this end we define $\mathtt{Vect}^n_+(B)$ to denote the space of oriented rank n vector bundles over a base space B, and we get an isomorphism

$$[S^{k-1}, \mathrm{GL}^+_n(\mathbb{R})] \to \mathtt{Vect}^n_+(S^k) = \mathtt{Vect}^n(S^k).$$

This in fact means that every real vector bundle over S^k (with $k > 1$) is oriented and has two choices of orientation [Hat03, p. 25]. Recall that the Gram-Schmidt orthogonalization procedure gives a deformation retraction

$$\mathrm{GL}_n(\mathbb{R}) \xrightarrow{\sim} O(n).$$

In particular, it gives a deformation retraction $\mathrm{GL}^+_n(\mathbb{R}) \xrightarrow{\sim} \mathrm{SO}(n)$ reducing matrices of positive nonzero determinant to matrices of determinant $+1$. Hence we get a bijection

$$[S^{k-1}, \mathrm{SO}(n)] \to [S^{k-1}, \mathrm{GL}^+_n(\mathbb{R})] = \mathtt{Vect}^n_+(S^k).$$

And as above, we see that $\mathtt{Vect}^+_n(S^k) = [S^k, B\mathrm{SO}(n)]$. A similar argument shows that Gram-Schmidt gives a deformation retract $\mathrm{GL}_n(\mathbb{C}) \to U(n)$ to unitary matrices.

At this stage we should be asking two questions:

(1) What are these spaces $\mathrm{BGL}_n(\mathbb{R})$, $B\mathrm{SO}(n)$, $\mathrm{BGL}_n(\mathbb{C})$, $\mathrm{BU}(n)$?
(2) Can we say that $\mathtt{Vect}^n_{\mathbb{C}}(-) = [-, \mathrm{BGL}_n(\mathbb{C})]$ etc for spaces other than spheres?

2.1. **Universal bundles.**

Definition 2.2. The *Grassmannian* $\mathrm{Gr}_k(\mathbb{R}^n)$ is the moduli space of k-planes in $\mathbb{R}^n$. If we want no constraints on where the k-planes are permitted to live, we could just think of $\mathrm{Gr}_k(\mathbb{R}^\infty)$ as the moduli space of k-planes, allowed to move freely around in any number of dimensions.

A rank n vector bundle on a space X assigns to each $x \in X$ an n-dimensional vector space, and does so in a continuous way. Seeing this definition, we might imagine that a vector bundle could be determined by a map of X into a Grassmannian. That is, we might want to describe a vector bundle as a map

$$X \to \mathrm{Gr}_n(\mathbb{R}^\infty).$$

Indeed, this is the case, and moreover, homotopic maps give isomorphic vector bundles. Thus we obtain the following theorem.

Theorem 2.3. Let X be paracompact. Then we obtain an isomorphism

$$[X, \mathrm{Gr}_n(\mathbb{R}^\infty)] \cong \mathtt{Vect}_n^{\mathbb{R}}(X).$$

The way this theorem is usually expressed is as follows: let $E_n(\mathbb{R}^\infty)$ denote the moduli space of marked n-planes, that is, planes with a point in them:

$$E_n(\mathbb{R}^\infty) = \{(L, v) \in \mathrm{Gr}_n(\mathbb{R}^\infty) \times \mathbb{R}^n \ : \ v \in L\}.$$

This comes equipped with a natural projection down to the Grassmannian

$$\begin{aligned} \gamma_n : E_n(\mathbb{R}^\infty) &\twoheadrightarrow \mathrm{Gr}_n(\mathbb{R}^\infty) \\ (L, v) &\mapsto L, \end{aligned}$$

obtained by forgetting the marked point. Moreover, it is clear that this is a rank n vector bundle.

Now for any homotopy class of map $f : X \to \mathrm{Gr}_n(\mathbb{R}^\infty)$ we can construct a rank n bundle on X by taking the pullback:

$$\begin{array}{ccc} f^*E_n & \longrightarrow & E_n(\mathbb{R}^\infty) \\ \big\downarrow & \lrcorner & \big\downarrow \gamma_n \\ X & \xrightarrow[f]{} & \mathrm{Gr}_n(\mathbb{R}^\infty). \end{array}$$

The map $f \mapsto f^*(E_n)$ is precisely the bijection $[X, \mathrm{Gr}_n(\mathbb{R}^\infty)] = \mathtt{Vect}_n^{\mathbb{R}}(X)$. This is a natural isomorphism of functors.

Theorem 2.4. Similarly for complex vector bundles, for paracompact X we have

$$\mathtt{Vect}_{\mathbb{C}}^n(X) = [X, \mathrm{Gr}_n(\mathbb{C}^\infty)].$$

For oriented vector bundles, we have an oriented $\widetilde{\mathrm{Gr}}_n(\mathbb{R}^\infty)$, which comes equipped with a map $\widetilde{\mathrm{Gr}}_n(\mathbb{R}^\infty) \twoheadrightarrow \mathrm{Gr}_n(\mathbb{R}^\infty)$. Moreover, this is a universal cover of the Grassmannian, since it is simply connected. We can see this since $\pi_1(\widetilde{\mathrm{Gr}}_n(\mathbb{R}^\infty)) = [S^1, \widetilde{\mathrm{Gr}}_n(\mathbb{R}^\infty)] = \mathtt{Vect}_n^+(S^1) = 0$.

Finally, we will align this with our discussion of vector bundles from earlier. The following are homotopy equivalent:

$$\begin{aligned} \mathrm{BGL}_n(\mathbb{R}) &\simeq \mathrm{BO}(n) \simeq \mathrm{Gr}_n(\mathbb{R}^\infty), \\ \mathrm{BGL}_n(\mathbb{C}) &\simeq \mathrm{BU}(n) \simeq \mathrm{Gr}_n(\mathbb{C}^\infty), \\ \mathrm{BGL}_n^+(\mathbb{R}) &\simeq \mathrm{BSO}(n) \simeq \widetilde{\mathrm{Gr}}_n(\mathbb{R}^\infty). \end{aligned}$$

In particular, we can see

$$\begin{aligned} \mathrm{BO}(1) &= \mathrm{Gr}_1(\mathbb{R}^\infty) = \mathbb{R}\mathrm{P}^\infty = K(\mathbb{Z}/2,1), \\ \mathrm{BU}(1) &= \mathrm{Gr}_1(\mathbb{C}^\infty) = \mathbb{C}\mathrm{P}^\infty = K(\mathbb{Z},2). \end{aligned}$$

Hence for any paracompact base space X we have that

$$\begin{aligned} \mathtt{Vect}_{\mathbb{R}}^n(X) &= [X, \mathrm{BO}(n)], \\ \mathtt{Vect}_{\mathbb{R}}^{n,+}(X) &= [X, \mathrm{BSO}(n)], \\ \mathtt{Vect}_{\mathbb{C}}^n(X) &- [X, \mathrm{BU}(n)]. \end{aligned}$$

In particular, we have that rank 1 bundles (that is, line bundles) on X are given by

$$\begin{aligned} \mathtt{Vect}_{\mathbb{R}}^1(X) &= [X, K(\mathbb{Z}/2,1)] = H^1(X;\mathbb{Z}/2), \\ \mathtt{Vect}_{\mathbb{C}}^1(X) &= [X, K(\mathbb{Z},2)] = H^2(X;\mathbb{Z}). \end{aligned}$$

Remark 2.5. For BSU it is not quite as simple. As any complex plane comes equipped with a canonical orientation, we end up with a principal S^1-bundle

$$S^1 \hookrightarrow \mathrm{BSU}(n) \twoheadrightarrow \widehat{\mathrm{Gr}}_n(\mathbb{C}^\infty).$$

Proposition 2.6. Let M be a real n-manifold without boundary. Then its tangent bundle corresponds to a map $f : M \to \mathrm{BO}(n)$, and we have that M is parallelizable if and only if f is null-homotopic.

3. Characteristic classes

Example 3.1. In order to check if $E \to B$ is orientable, we can check the following: supposing that B is path-connected, we can lift every loop in B to E, which induces a transformation of the fiber, and we can see whether this transformation is orientation-preserving. Explicitly, lifting a loop gives an element of $\mathrm{GL}_n(\mathbb{R})$ and we can check whether its determinant is positive or negative. Hence we get a composite group homomorphism

$$\pi_1(B) \to \mathrm{GL}_n(\mathbb{R}) \to \mathbb{Z}/2.$$

By the universal property of abelianization, this map factors through $\pi_1^{\mathrm{ab}}(B) = H_1(B;\mathbb{Z})$. Applying UC, we see that

$$0 \to \mathrm{Ext}(H_0(B;\mathbb{Z}),\mathbb{Z}/2) \to H^1(B;\mathbb{Z}/2) \to \mathrm{Hom}(H_1(B;\mathbb{Z});\mathbb{Z}/2),$$

and since $\mathrm{Ext}(\mathbb{Z}, \mathbb{Z}/2) = 0$[1], we have that $H^1(B; \mathbb{Z}/2) \cong \mathrm{Hom}(H_1(B; \mathbb{Z}), \mathbb{Z}/2)$. Thus the procedure above (of lifting loops to see if they preserve the orientation of the fiber and then abelianizing) corresponds to an element of $H^1(B; \mathbb{Z}/2)$. This element is denoted by $w_1(E)$, called the *first Stiefel Whitney class* [Hat03, p.78]. We have that

$$E \to B \text{ is orientable} \iff w_1(E) \in H^1(B; \mathbb{Z}/2) \text{ vanishes.}$$

A categorical description of vector bundles may be taken as follows: for any cohomology theory $h^*(-)$, a *characteristic class* of degree q for rank n bundles is a map (for every base space B)

$$\begin{aligned} c : \mathtt{Vect}_n(B) &\to k^q(B) \\ \xi &\mapsto c(\xi) \in k^q(B). \end{aligned}$$

For real vector bundles it sends $[B, \mathrm{BO}(n)] \to k^q(B)$. More explicitly, it is a natural transformation

$$c : [-, \mathrm{BO}(n)] \Rightarrow k^q(-).$$

By the Yoneda lemma, we therefore get a bijection

$$\begin{aligned} \{\text{characteristic classes of } n\text{-plane bundles}\} &\longleftrightarrow k^*(\mathrm{BO}(n)) \\ c &\longmapsto c(\gamma_n). \end{aligned}$$

So determining characteristic classes should reduce to the computation of cohomology theories of Grassmannians. Before we do this, let's motivate why we might care about characteristic classes.

Let $p : E \to B$ be a rank n vector bundle, and let c be a characteristic class in degree q of n-plane bundles. Recall by the classification of vector bundles, we have that $p : E \to B$ is determined by pulling back the universal bundle via a map $f : B \to \mathrm{Gr}_n$, that is,

$$\begin{array}{ccc} E = f^*(E_n) & \longrightarrow & E_n \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle \gamma_n} \\ B & \xrightarrow[f]{} & \mathrm{Gr}_n. \end{array}$$

Then by the naturality of the characteristic class c, we see that the induced map of cohomology sends one characteristic class to another, that is,

$$\begin{aligned} f^* : k^q(E) &\to k^q(B) \\ c(\gamma_n) &\mapsto f^*c(\gamma_n) = c(p) = c(f^*\gamma_n). \end{aligned}$$

[1]Since $\mathbb{Z}$ is free and hence projective.

This last equality is the important property of characteristic classes:

$$c(f^*\gamma_n) = f^*c(\gamma_n).$$

Since every bundle is a pullback of the universal bundle, we see that characteristic classes on all base spaces are just pulled back from characteristic classes evaluated on the universal bundle. So now we know how to find characteristic classes, and how to evaluate them on arbitrary spaces.

The main type of bundles are as follows:

Symbol	Name	Type of bundle	Degree and coefficients
w_i	*Stiefel-Whitney classes*	real vector bundle	$w_i(E) \in H^i(B; \mathbb{Z}/2)$
c_i	*Chern classes*	complex vector bundle	$c_i(E) \in H^{2i}(B; \mathbb{Z})$
p_i	*Pontryagin classes*	real vector bundle	$p_i(E) \in H^{4i}(B; \mathbb{Z})$
e	*Euler class*	oriented rank n bundle	$e(E) \in H^n(B; \mathbb{Z})$.

We now would like to compute the cohomology of the infinite Grassmannians. Let E_n denote the universal cover of the infinite Grassmannian, when the field is already specified. We have that the cohomology of the Grassmannian is given by

$$H^*(\mathrm{Gr}_n(\mathbb{R}^\infty); \mathbb{Z}/2) = \mathbb{Z}/2[w_1(E_n), \dots, w_n(E_n)],$$
$$H^*(\mathrm{Gr}_n(\mathbb{C}^\infty); \mathbb{Z}) = \mathbb{Z}[c_1(E_n), \dots, c_n(E_n)].$$

3.1. **Stiefel-Whitney Classes. The four axioms**:

(1) For each ξ vector bundle the *Stiefel-Whitney classes* are

$$w_i(\xi) \in H^i(B; \mathbb{Z}/2), \ i = 0, 1, \dots$$

Moreover we have that $w_0(\xi) = 1 \in H^0(B; \mathbb{Z}/2)$, and if ξ is an n-plane bundle then $w_i(\xi) = 0$ for all $i > n$.

(2) *(naturality)* For $f : \xi \to \eta$ we have that $w_i(\xi) = f^*w_i(\eta)$.

(3) *(Whitney Product Theorem)* if ξ and η are both bundles over B then

$$w_k(\xi \oplus \eta) = \sum_{i=0}^{k} w_i(\xi) \cup w_{k-i}(\eta).$$

(4) For γ_1^1 canonical bundle over $\mathbb{R}\mathrm{P}^1$, we have that $w_1(\gamma_1^1) \neq 0$.

Proposition 3.2. If ε is trivial, then $w_i(\varepsilon) = 0$ for all $i > 0$.

Proposition 3.3. We have that $w_1(\xi) = 0$ if and only if ξ is orientable. In particular for a manifold M we have that M is orientable if and only if $w_1(TM) = 0$.

Let $w := \sum_i w_i \in H^*(B; \mathbb{Z}/2)$ denote the *total Stiefel-Whitney class*. Let $\overline{w}$ denote its inverse, so that $w\overline{w} = 1 \in H^*(B; \mathbb{Z}/2)$.

Proposition 3.4. *(Whitney Product Theorem)* We can restate it as:

$$w(\xi \oplus \eta) = w(\xi)w(\eta).$$

Theorem 3.5. *(Whitney Duality Theorem)* Let M be a manifold in Euclidean space, and let τ_M denote its tangent bundle and ν_M its normal bundle. Then

$$w_i(\nu_M) = \overline{w}_i(\tau_M).$$

Example 3.6. We have that $w_1 : \mathtt{Vect}^1_{\mathbb{R}}(X) \to H^1(X : \mathbb{Z}/2)$ is an isomorphism if X has the homotopy type of a CW complex [Hat03, p. 3.10].

Proposition 3.7. A vector bundle $E \to X$ is orientable if and only if $w_1(E) = 0$.

Proof. Supposing X is path-connected, there are natural isomorphisms

$$H^1(X; \mathbb{Z}/2) \cong \mathrm{Hom}(H_1(X); \mathbb{Z}/2) \cong \mathrm{Hom}(\pi_1(X); \mathbb{Z}/2).$$

The first comes by UC (since $\mathbb{Z}/2$ is a field) and the second by the universal property of abelianization. □

Definition 3.8. Let M be an n-manifold, and let $r_1, \dots, r_n \geq 0$ so that $\sum_{k=1}^n k \cdot r_k = n$. Then we define the *Stiefel-Whitney number* in $\mathbb{Z}/2$ as

$$\langle w_1(\tau_M)^{r_1} \cdots w_n(\tau_M)^{r_n}, [\mu_M] \rangle = w_1^{r_1} \cdots w_n^{r_n}[M].$$

Proposition 3.9. Two smooth closed manifolds have the same Stiefel-Whitney numbers if and only if they lie in the same cobordism class [MS74, p. 53].

Theorem 3.10. *(Thom)* All the Stiefel-Whitney *numbers* (not classes) of an n-manifold M are zero if and only if $M = \partial B$ is the boundary of a smooth compact $(n+1)$-manifold.

3.2. **Euler class and Thom spaces.** Let $\xi : E \to B$ be a k-plane bundle equipped with a Euclidean metric (that is, a principal $O(n)$-bundle). We can define the *Thom space* of this bundle as follows: let $D(E) = \{v \in E \ : \ |v| \leq 1\}$ be the disk subbundle, and $S(E) = \{v \in E \ : \ |v| = 1\}$ be the sphere subbundle. Then

$$\mathrm{Th}(\xi) = \mathrm{Th}(E) := D(E)/S(E).$$

The entire sphere bundle is collapsed to single point, which we will denote by t_0. We might think about this as a closed unit disk bundle, compactified at every point in the fiber in a coherent way. Moreover, since $S(E) \rightarrowtail D(E)$ is an inclusion of cell complexes, we can think of $\mathrm{Th}(E)$ as a homotopy cofiber.

Let E_0 denote the space of nonzero vectors in E. Then if ξ has a Euclidean metric, we can deformation retract E to $D(E)$ and E_0 to $S(E)$. Hence we might think of $\mathrm{Th}(E)$ again as $\mathrm{hocofib}(E_0 \hookrightarrow E)$.

Definition 3.11. We define the *fundamental class* of the Thom space $\mathrm{Th}(E)$ of an n-plane bundle as the unique cohomology class $u \in \widetilde{H}^n(\mathrm{Th}(E); \mathbb{Z}/2) = H^n(E, E_0; \mathbb{Z}/2)$ so that its restriction to $\widetilde{H}^n(F, F_0; \mathbb{Z}/2)$ is nonzero for every fiber.

Theorem 3.12. Again for an n-plane bundle $\xi : E \to B$, one has the *Thom isomorphism* with $\mathbb{Z}/2$ coefficients, which is the composition of the two isomorphisms

$$H^k(B;\mathbb{Z}/2) \xrightarrow{\pi^*} H^k(E;\mathbb{Z}/2) \xrightarrow{-\cup u} H^{k+n}(E, E_0;\mathbb{Z}/2) = \widetilde{H}^{k+n}(\mathrm{Th}(E);\mathbb{Z}/2).$$

This can be found in [MS74, p. 8.2].

Theorem 3.13. *(Thom's Isomorphism Theorem)* The correspondence

$$\begin{aligned} H^j(E;R) &\to \widetilde{H}^{j+n}(\mathrm{Th}(E);R) \\ y &\mapsto y \cup u, \end{aligned}$$

is an isomorphism for each $j \in \mathbb{Z}$ and for any coefficient ring R [MS74, pp. 10.2, 10.4].

Corollary 3.14. Taking $j < 0$ we see that

$$\widetilde{H}^i(\mathrm{Th}(E);R) = 0, \quad i < n.$$

Via the cap product, we get an analogous statement for homology.

Proposition 3.15. The correspondence

$$\begin{aligned} \widetilde{H}_{n+i}(\mathrm{Th}(E);R) &\to H_i(E;R) \\ \eta &\mapsto u \cap \eta, \end{aligned}$$

is an isomorphism for all i and for all R [MS74, p. 10.7]. If furthermore we have that $\xi : E \to B$ is oriented, then for integral coefficients we have

$$H_i(B;\mathbb{Z}) \cong \widetilde{H}_{n+i}(\mathrm{Th}(E);\mathbb{Z})$$

for all i [MS74, p. 18.2].[2]

Returning to the isomorphism on cohomology, in the case where we are taking integral coefficients, we have a *Thom isomorphism*:

$$H^k(B;\mathbb{Z}) \xrightarrow{\pi^*} H^k(E;\mathbb{Z}) \xrightarrow{-\cup u} \widetilde{H}^{k+n}(\mathrm{Th}(E);\mathbb{Z}).$$

Definition 3.16. The *Euler class* of an oriented n-plane bundle $\xi : E \to B$ is the cohomology class

$$e(\xi) \in H^n(B;\mathbb{Z}),$$

corresponding to the fundamental class $u|_E$ under the canonical isomorphism $\pi^* : H^n(B;\mathbb{Z}) \to H^n(E;\mathbb{Z})$.

Properties of the Euler class

(1) If the orientation of ξ is reversed then $e(\xi)$ changes sign

[2] In [MS74] they refer to the homology relative a point, but we can recall from Hatcher, say, that $\widetilde{H}_i(X) \cong H_i(X, x_0)$. The proof of the isomorphism $H_i(B;\mathbb{Z}) \cong \widetilde{H}_{n+i}(\mathrm{Th}(E);\mathbb{Z})$ can be seen via an easy excision argument.

(2) We have that $e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$ and that $e(\xi \times \xi') = e(\xi) \times e(\xi')$.
(3) We have that the natural homomorphism sends the Euler class to the top Stiefel-Whitney class:

$$\begin{aligned} H^n(B;\mathbb{Z}) &\to H^n(B;\mathbb{Z}/2) \\ e(\xi) &\mapsto w_n(\xi). \end{aligned}$$

(4) If an oriented vector bundle ξ has a nowhere zero cross-section, then $e(\xi) = 0$.

Fun fact 3.17. The *Euler characteristic* of a smooth manifold M is its *Euler number*, that is, the Euler class evaluated on the fundamental class:

$$\chi(M) = e(TM)([M]).$$

Example 3.18. The tangent bundle $TS^6 \to S^6$ admits no proper oriented sub-bundles.

Proof. Since S^6 is a smooth manifold, it can be embedded by Whitney (TODO) which gives it a metric, and induces a metric on the tangent space. Hence we can talk about orthogonality of sub-bundles.

Say E were a proper sub-bundle of TS^6 of dimension $k < 6$. Then we have that $e(E) \in H^k(S^6;\mathbb{Z}) = 0$, hence $e(E) = 0$. However we note that

$$0 = e(E) \cup e(E^\perp) = e(E \oplus E^\perp) = e(TS^6).$$

However this would imply that $\chi(S^6) = 0$, since we have that $\chi(S^6) = 1 + (-1)^6 = 2$.

□

Finally, one should discuss the *Gysin sequence*, an important long exact sequence in homology:

Definition 3.19. Let $\xi : E \to B$ be an oriented n-plane bundle. Let $\pi_0 : E_0 \to B$ be the subbundle of nonzero vectors in E, and let $e = e(\xi) \in H^n(B;\mathbb{Z})$ denote the Euler class. Then we have a long exact sequence, called the *Gysin sequence*

$$\cdots \to H^i(B;\mathbb{Z}) \xrightarrow{-\cup e} H^{i+n}(B;\mathbb{Z}) \xrightarrow{\pi_0^*} H^{i+n}(E_0;\mathbb{Z}) \to H^{i+1}(B;\mathbb{Z}) \xrightarrow{-\cup e} \cdots$$

This is obtained via the long exact sequence of the pair (E, E_0) and the Thom isomorphism [MS74, p. 12.2].

3.3. Digression: Tubular neighborhoods.

Theorem 3.20. *(Tubular neighborhood theorem)* Let $M^n \hookrightarrow A^{n+k}$ be a smooth manifold which is smoothly embedded in a Riemannian manifold. Then there is an open neighborhood $U \supseteq M$ in A, which is diffeomorphic to the total space E of the normal bundle ν^k of M in A [MS74, p. 11.1].

If $M \subseteq A$ is closed, then we can equate the cohomology of the Thom space of the normal bundle with the local cohomology of M in A [MS74, p. 11.2]:

$$\widetilde{H}^*(\mathrm{Th}(\nu^k); R) \cong H^*(A|M; R).$$

Moreover this isomorphism does not depend upon a choice of Riemannian metric for A.

Let $u' \in H^k(A|M; \mathbb{Z}/2)$ denote the image of the fundamental cohomology class $u \in H^k(\mathrm{Th}(\nu^k); \mathbb{Z}/2)$ under the isomorphism above. We call this the *dual cohomology class* to M in codimension k. Then we have a number of interesting results:

Theorem 3.21. If $M \subseteq A$ is closed and embedded, then the composition

$$\begin{aligned} H^k(A|M; \mathbb{Z}/2) \to H^k(A; \mathbb{Z}/2) &\to H^k(M; \mathbb{Z}/2) \\ u' &\longmapsto w_k(\nu^k), \end{aligned}$$

sends u' to the top Stiefel-Whitney class of the normal bundle. If ν^k is oriented, then u' is mapped to the Euler class [MS74, p. 11.3]

$$\begin{aligned} H^k(A|M; \mathbb{Z}) \to H^k(A; \mathbb{Z}) &\to H^k(M; \mathbb{Z}) \\ u' &\longmapsto e(\nu^k). \end{aligned}$$

Hence if $u'|_A = 0$ then $w_k(\nu^k)$ or $e(\nu^k)$ must be zero (depending on the existence of an orientation). As an interesting corollary, we see

Corollary 3.22. If $M^n \hookrightarrow \mathbb{R}^{n+k}$ is smoothly embedded, then $u'|_{\mathbb{R}^{n+k}} \in H^k(\mathbb{R}^{n+k}) = 0$, and then $w_k(\nu^k) = 0$. In the oriented case, we have $e(\nu^k) = 0$ [MS74, p. 11.4].

Invoking Whitney duality, we have $w_k(\nu^k) = \overline{w}_k(\tau_M)$, hence *if $\overline{w}_k(\tau_M) \neq 0$ then M cannot be smoothly embedded as a closed subset in $\mathbb{R}^{n+k}$.*

This statement has deep impacts, which we may see in the following examples.

Example 3.23 (Immersing real projective space)**.** Recall that

$$H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2) = \frac{(\mathbb{Z}/2)[a]}{(a^{n+1})}, \quad |a| = 1.$$

It is a theorem [MS74, Theorem 4.5] that $w(\tau_{\mathbb{R}\mathrm{P}^n}) = (1+a)^{n+1}$ where a is the generator of $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2)$ as above.

Suppose $\mathbb{R}\mathrm{P}^n \hookrightarrow \mathbb{R}^{n+k}$ is an immersion. Then by Whitney duality, we have that

$$w_i(\nu_{\mathbb{R}\mathrm{P}^n}) = \overline{w}_i(\tau_{\mathbb{R}\mathrm{P}^n}).$$

Consider the example of $\mathbb{R}\mathrm{P}^9$. Then since we are in characteristic 2, we compute

$$\begin{aligned} w(\tau_{\mathbb{R}\mathrm{P}^9}) &= (1+a)^{10} = (1+a)^{2^3}(1+a)^2 = (1+a^8)(1+a^2) \\ &= 1 + a^2 + a^8. \end{aligned}$$

We compute the normal bundle as:

$$w(\nu_{\mathbb{RP}^9}) = \overline{w}(\tau_{\mathbb{RP}^9}) = 1 + a^2 + a^4 + a^6.$$

Thus if $\mathbb{RP}^9 \hookrightarrow \mathbb{R}^{9+k}$ is an immersion, then we must have $k \geq 6$.

Corollary 3.24. If $\mathbb{RP}^{2^r} \hookrightarrow \mathbb{R}^{2^r+k}$ is an immersion, then $k \geq 2^r - 1$.

Example 3.25 (Embedding real projective space)**.** By [MS74, Corollary 11.4] if $M^n \hookrightarrow \mathbb{R}^{n+k}$ is an embedding, then $w_k(\nu^k) = 0$. In the oriented case, $e(\nu^k) = 0$.

Corollary 3.26. We cannot have a smooth embedding $\mathbb{RP}^2 \hookrightarrow \mathbb{R}^3$.

Proof. Suppose towards a contradiction this is true. Then we compute its Stiefel-Whitney class as

$$w(\tau_{\mathbb{RP}^2}) = (1 + a)^3 = (1 + a^2)(1 + a) = 1 + a + a^2.$$

Its inverse Stiefel Whitney class is then

$$w(\nu_{\mathbb{RP}^2}) = \overline{w}(\tau_{\mathbb{RP}^2}) = \frac{1}{1 + a + a^2} = \frac{1 + a^3}{1 + a + a^2} = (1 + a).$$

But by [MS74, p. 11.4], we have that $w_1(\nu^1) = 0$ if $\mathbb{RP}^2 \hookrightarrow \mathbb{RP}^3$ is a smooth embedding. However this contradicts the computation above, since we saw $w_1(\nu) = a \neq 0$. □

We additionally get a relative version of Poincaré, where we can see what happens to the dual cohomology class of a compact, smoothly embedded oriented submanifold.

Theorem 3.27. *(Relative Poincaré)* Let $i : M^n \hookrightarrow A^{n+p}$ be a smooth embedding of compact, oriented manifolds. Then the Poincaré duality isomorphism sends $u'|_A$ to $(-1)^{nk} i_*([M])$:

$$\begin{aligned} - \cap [A] : H^k(A) &\xrightarrow{\cong} H_n(A) \\ u'\big|_A &\mapsto (-1)^{nk} i_*[M]. \end{aligned}$$

This may be found in [MS74, pp. 11–C].

As a result of relative Poincaré, we have the following example.

Example 3.28. Let M be a closed orientable n-manifold. Then every codimension 1 homology class in M is represented by a submanifold of codimension 1.

Proof. Recall for Brown representability, the isomorphism $[X, K(A, n)] \cong H^n(X; A)$ is obtained via $f \mapsto f^*u$, where $u \in H^n(K(A, n); n)$ is the *fundamental class*, corresponding to $\mathrm{id} \in [K(A, n), K(A, n)]$.

Let $v \in H^1(S^1;\mathbb{Z})$ be the fundamental class. Then by Poincaré duality, we have a string of isomorphisms

$$\begin{aligned}[M,S^1] \to H^1(M;\mathbb{Z}) \to H_{n-1}(M;\mathbb{Z})\\ f \mapsto f^*v \mapsto [M]\cap f^*v.\end{aligned}$$

For such an f, we might homotope it to be smooth, and then apply Sard's Theorem and the preimage theorem to find a regular value $x \in S^1$ so that $Z := f^{-1}(x)$ is a codimension 1 submanifold.

Let $i : Z \hookrightarrow M$ be the inclusion of the submanifold. Then we would like to send $f \in [M,S^1]$ to $i_*[Z] \in H_{n-1}(M;\mathbb{Z})$ and to see that this is an isomorphism. In order to do this, it suffices to establish the equality

$$i_*[f^{-1}(x)] = [M]\cap f^*(v).$$

By Theorem 3.27, it will suffice to see that $f^*(v)$ is the dual cohomology class of $Z \subseteq M$.

Now let v be a generator for $H^1(S^1;\mathbb{Z})$. Then our string of isomorphisms sends

$$\begin{aligned}[M,S^1] \to H^1(M;\mathbb{Z}) \to H_{n-1}(M;\mathbb{Z})\\ f \mapsto f^*(v) \mapsto [M]\cap f^*(v).\end{aligned}$$

Now let's recall the following

Relative Poincaré: Let $M^n \subseteq A^p$ be compact oriented manifolds, with a smooth embedding $i : M \to A$. Let $u \in H^k(\mathrm{Th}(\nu^k);\mathbb{Z}/2)$ denote the *fundamental cohomology class* for the Thom space of the normal bundle of M in A, and let u' denote the corresponding element in $H^k(A|M;\mathbb{Z}/2)$ under the isomorphism. Then the Poincaré duality map sends

$$\begin{aligned}H^k(A) \to H_n(A)\\ u'\big|_A \mapsto (-1)^{nk} i_*[M].\end{aligned}$$

[MS74, pp. 11–C].

Now let $Z = f^{-1}(x)$ denote our submanifold. By naturality of the Thom space, one has that $\mathrm{Th}(\nu_Z) = f^*\mathrm{Th}(\nu_{\{x\}})$ is the pullback of the Thom space of the normal bundle of x in S^1, which is (SOMEHOW) the generator of $H^1(S^1)$.

Thus under Poincaré duality, one has that $i_*f^{-1}(x)$ corresponds to the map f in $[M,S^1]$.

□

3.4. **Chern classes.** For a *complex vector bundle* $E \to X$, we have sequence of characteristic classes $c_i(E) \in H^{2i}(B;\mathbb{Z})$ satisfying the following axioms:

(1) $c_0(E) = 1$
(2) naturality

(3) Whitney sum formula, we have $c(E \oplus F) = c(E) \cup c(F)$, that is,

$$c_k(E \oplus F) = \sum_{i=0}^{k} c_i(E) \cup c_{k-i}(F).$$

(4) for the canonical bundle $E \to \mathbb{C}\mathrm{P}^\infty$, one has that $c_1(E)$ is a generator for $H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$.

Constructing the Chern classes: Consider the embedding map

$$\prod_{i=1}^{n} U(1) \hookrightarrow U(n),$$

whose image is $n \times n$ diagonal complex matrices. Since diagonal matrices lie in the center

This induces a map on classifying spaces

$$B\left(\prod_{i=1}^{n} U(1)\right) = \prod_{i=1}^{n} \mathbb{C}\mathrm{P}^\infty \to \mathrm{BU}(n).$$

By the Künneth theorem, one has that

$$H^*\left(\prod_{i=1}^{n} \mathbb{C}\mathrm{P}^\infty; \mathbb{Z}\right) = \bigotimes_{i=1}^{n} H^*(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) = \mathbb{Z}[\beta_1, \dots, \beta_n],$$

with $|\beta_i| = 2$. This comes with an action of the symmetric group by permuting the variables. Letting σ_j denote the jth symmetric polynomial in the β_i's, one has that

$$H^*((\mathbb{C}\mathrm{P}^\infty)^n; \mathbb{Z})^{\Sigma_n} = \mathbb{Z}[\beta_1, \dots, \beta_n]^{\Sigma_n} = \mathbb{Z}[\sigma_1, \dots, \sigma_n].$$

We claim that there is a factorization

$$\begin{array}{ccc} H^*(\mathrm{BU}(n); \mathbb{Z}) & \longrightarrow & H^*((\mathbb{C}\mathrm{P}^\infty)^n; \mathbb{Z}) \\ & \overset{f}{\searrow} & \uparrow \\ & & H^*((\mathbb{C}\mathrm{P}^\infty)^n; \mathbb{Z})^{\Sigma_n} \end{array}$$

Finally, we define $c_i := f^*\sigma_i$. Then one has

$$H^*(\mathrm{Gr}_n(\mathbb{C}^\infty); \mathbb{Z}) = \mathbb{Z}[c_1(\gamma^n), \dots, c_n(\gamma^n)].$$

Proposition 3.29. We have the following additional properties:

(1) *additivity*: for an exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0,$$

we have $c(E) = c(E') \cup c(E'')$.

(2) If E is a rank n complex bundle, then $c_n(E) = e(E)$ is the Euler class of the underlying real bundle.c

(3) if ε^k is a trivial complex k-plane bundle, then one has

$$c(\omega \oplus \varepsilon^k) = c(\omega).$$

(4) we have

$$c(\tau_{\mathbb{C}\mathrm{P}^n}) = (1+a)^{n+1},$$

where a is a generator of $H^2(\mathbb{C}\mathrm{P}^n;\mathbb{Z})$.

4. Principal bundles

Let G be a topological group, and let $p : Y \to B$ be a vector bundle with fiber G. We say this is a *principal G-bundle* if we have a right action of G on Y which preserves fibers and acts freely and transitively on the fibers.[3] In this case, we will have that $Y/G \cong B$.

Definition 4.1. A *universal principal G-bundle* is one of the form $Y \to Y/G$ where Y is contractible.

Any equivariant morphism between the total spaces is a *principal bundle morphism*.

As above, we have a representability result. Let $\mathcal{PG}(B)$ denote the set of equivalence classes of principal G-bundles over a base space B. Then for any universal principal G-bundle $Y \to Y/G$ there is a natural isomorphism

$$[-, Y/G] \Rightarrow \mathcal{PG}(-).$$

5. Associated bundles

Let $p : E \to B$ be a principal G, bundle, and suppose that G acts on a space F via $\rho : G \to \mathrm{Aut}(F)$. Then we can define the balanced product

$$E \times_G F := E \times F/\sim,$$

where $(e, gf) \sim (ge, f)$. This is equipped with a projection map $\pi_F : E \times_G F \to B$. This defines a fiber bundle with fiber F. We call this the *associated fiber bundle* to the principal bundle. The fibers of this bundle have structure group $\mathrm{Aut}(F)$.

Conversely, given any fiber bundle $F \hookrightarrow E \twoheadrightarrow B$ with structure group $\mathrm{Aut}(F)$, there exists a principal $\mathrm{Aut}(F)$-bundle P so that $E = P \times_G F$. We call this principal bundle the *frame bundle*. read more here, p.9

Example 5.1. Classify all associated S^2-bundles over S^2.

[3] Recall *free* means the only group element fixing an element y is the identity element ($y \cdot g = y \Rightarrow g = e$). *Transitive* means that for all y, y' there exists a (unique) g so that $y \cdot g = y'$. This is unique since the action is free.

Proof. Given a fiber bundle $S^2 \hookrightarrow E \twoheadrightarrow S^2$, its associated bundle is a principal $\mathrm{Aut}(S^2)$-bundle over S^2. We may check that $\mathrm{Aut}(S^2)$ is a subgroup of $\mathrm{GL}_3(\mathbb{R})$ which preserves length, i.e. it is $O(3)$. Hence it suffices to classify principal $O(3)$-bundles over S^2. This is done by noting that $O(3) = \mathrm{SO}(3) \amalg \mathrm{SO}(3)$, and hence we have:

$$[S^2, \mathrm{BO}(3)] = [S^1, \Omega\mathrm{BO}(3)] = \pi_1(O(3)) = \pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2.$$

Hence we have the trivial bundle $S^2 \times S^2$ and we have one more bundle, which turns out to be the connected sum $\mathbb{C}\mathrm{P}^2 \# \mathbb{C}\mathrm{P}^2 = \mathrm{Bl}_0\mathbb{P}^2$. □

6. Reducing the structure group

We have the following table of reduction:

Real Vector Bundles:

Group reduction	*Structure on total space*
$\mathrm{GL}_n(\mathbb{R}) \rightsquigarrow \mathrm{GL}_n^+(\mathbb{R})$	Orientation
$\mathrm{GL}_n^+(\mathbb{R}) \rightsquigarrow \mathrm{SL}_n(\mathbb{R})$	No obstruction (just need orientability)
$\mathrm{GL}_n(\mathbb{R}) \rightsquigarrow O(n)$	Riemannian metric

For complex bundles the story is similar:

Complex Vector Bundles:

Group reduction	*Structure on total space*
$\mathrm{GL}_n(\mathbb{C}) \rightsquigarrow U(n)$	Hermitian metric

References

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[MS74] John W. Milnor and James D. Stasheff. *Characteristic classes*. Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J.; University of Tokyo Press, Tokyo, 1974, pp. vii+331.