THE ALGEBRAIC VECTOR BUNDLE PROBLEM

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Today I want to talk about a fascinating problem at the intersection of algebra and topology. The rough imprecise formulation of it is as follows:

Which vector bundles can be described algebraically?

In order to explain what we mean here, let's go back in time a bit. A good starting point as any is the Weil conjectures, as laid out by André Weil in 1949. The rough formulation of the Weil conjecture deals with *zeta functions* $\zeta(X, s)$ associated to a projective algebraic variety X over a finite field. The formulation talks about generating functions, functional equations akin to the Riemann–Zeta function, and the "Riemann hypothesis" which investigates where the zeros lie in the complex plane. Perhaps the most interesting part of the conjectures for our purposes was the last, which we will state incredibly imprecisely

If X is "coming from" a complex manifold Y, then various parts of its zeta function can be seen in the Betti numbers of Y.

This part of the conjecture was huge. While the Weil conjectures broadly have had massive impacts, it was tremendously interesting to say that something like an elliptic curve over a finite field should have any connection to the Betti numbers of a manifold. Why in the world should something like algebraic topology, which hinges so strongly on our capacity to parametrize continuous changes, make an appearance in a discrete setting like a finite field?

This motivated a massive wave of research, which we owe much of modern algebraic geometry to. Weil gave very strong evidence that there should be some analogy of singular cohomology with integer coefficients in the world of algebraic geometry. This led to decades of work by Grothendieck, Serre, Artin, Deligne, and others, formulating a lot of the cohomology theories that we know and love today (étale, ℓ-adic, crystalline, motivic, etc.).

In 1955, Serre released FAC. It is not an exaggeration to say that this was one of the biggest papers ever released in algebra. It made precise the connections between vector bundles over spaces, and modules over rings. Let's look at a specific example.

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Recall that an *affine variety* is the zero locus of some polynomials over a ring, or more generally something that can locally be described using polynomials. For example we could look at the ring

$$A = \frac{\mathbb{R}[x,y]}{(x^2+y^2-1)}.$$

This lives honestly, genuinely, in the category of rings. However it is describing something topological! Namely we can take "real points," that is, the pairs of all $(x,y) \in \mathbb{R}^2$ that solve the equation that is being quotiented out. When we do this, we get the circle S^1.

Consider the matrix

$$\frac{1}{2}\begin{pmatrix} 1-x & y \\ y & 1-x \end{pmatrix} : A^2 \to A^2,$$

and let M be its image. Then M is a module over A, and it has a particular topological interpretation. Namely it is the Möbius band!

Let's recap what we did here. We started with a "vector bundle" over a ring, by which we mean a finitely generated projective module over a ring. By taking the associated real points, we obtained a vector bundle over S^1! What we want to say then is that the *topological vector bundle* defined by the Möbius band, admits an *algebraic interpretation.*

1. Topological versus algebraic vector bundles

For technical reasons it will be nicer to always work over the complex numbers. For the interest of keeping this talk even remotely accessible to the pure topologists, we will work always with affine varieties over $\mathbb{C}$.

Definition 1.1. An *affine variety* over $\mathbb{C}$ is the vanishing locus of some polynomials $f_1, \ldots, f_k$ in $\mathbb{C}[x_1, \ldots, x_n]$, for some k and n.

Example 1.2. The following are examples:

- $S^3 = V(x^2+y^2-1)$
- Tori are vanishing loci of elliptic curves
- The entirety of Euclidean space can be thought of as the vanishing locus of the zero polynomial

We will denote by $\texttt{Aff}_{\mathbb{C}}$ the category of complex affine varieties, and we will assume that we are working with smooth varieties.

Then we have a forgetful map

$$\begin{aligned} \texttt{Aff}_{\mathbb{C}} &\to \texttt{Top} \\ X &\mapsto X(\mathbb{C}). \end{aligned}$$

Theorem 1.3. (Andreotti–Frankel) If X is a smooth complex algebraic variety, then its underlying space has the homotopy type of a finite CW complex. (Proof uses Morse theory).

Definition 1.4. Let X be an affine variety of finite dimension, so that it is defined to be some ideal $I \trianglelefteq \mathbb{C}[x_1, \dots, x_n]$. Then we define an *algebraic vector bundle* on X to be a finitely generated projective module over the quotient ring $\frac{\mathbb{C}[x_1,\dots,x_n]}{I}$.

We denote by $V_r^{\text{alg}}(X)$ the set of isomorphism classes of algebraic vector bundles of rank r over X.

Remark 1.5. By forgetting the algebraic structure, we obtain a topological vector bundle over X. That is, there is a map

$$V_r^{\text{alg}}(X) \to V_r^{\text{top}}(X),$$

from rank r *algebraic* vector bundles on the *variety* X, to rank r *topological* vector bundles on the *space* X.

Example 1.6. Suppose that X is all of $\mathbb{C}^n$ (this corresponds to the zero ideal). Then every topological vector bundle over X is contractible (since X is contractible). Is the same true algebraically? That is, is every f.g. proj. module over $\mathbb{C}[x_1, \dots, x_n]$ trivial?

Serre's problem (1955) Are all finitely generated projective modules (vector bundles) over $k[x_1, \dots, x_n]$ trivial when k is a field?

Theorem 1.7. (Quillen–Suslin, 1976) Yes if k is a PID.

So for X equal to Euclidean space, we have that $V_r^{\text{alg}}(X) = V_r^{\text{top}}(X) = 0$. That is, the forgetful map is an isomorphism (algebraic and topological bundles agree over X).

The algebraization problem: Let X be any smooth complex affine variety. Which vector bundles admit an algebraic structure? That is, which bundles lie in the image of the forgetful map $V_r^{\text{alg}}(X) \to V_r^{\text{top}}(X)$?

This is still basically wide open, and some of the best mathematicians in the world are making steady progress on it. Let's see what types of machinery we could use to approach this problem.

2. The cycle class map

In 1958, Chevalley developed the *Chow groups*. The rough idea for Chow groups is that they encapsulate subvarieties in a similar way to how singular homology encapsulates submanifolds. That is, their elements are equivalence classes of *cycles*.

Let's keep working with the complex affine space. Let X be defined by the ring $A = \mathbb{C}[x_1, \dots, x_n]/I$. We can look at the prime ideals in this thing.

We get a filtration of the primes of A by the dimension of $A/\mathfrak{p}$. We can define

$$Z_i(X) := \bigoplus_{\mathfrak{p}\colon\ \dim(A/\mathfrak{p})=i} \mathbb{Z}.$$

This is the group of *i-cycles* on A.

Suppose that $\mathfrak{p}$ is a prime of height i. Then suppose that $x \in A - \mathfrak{p}$ (we think of x as a function defined on $A/\mathfrak{p}$). Then we get $[(A/\mathfrak{p})/x] \in Z_{*\leq i-1}(x)$.

This has an equivalence relation called rational equivalence, and when we mod out by it we get a homology theory! The resulting objects are called the *Chow groups*, and we denote them by $\mathrm{CH}_i(X) := Z_i(X)/\sim$.

Remember how in topology, submanifolds represented homology classes? The same thing is happening here, but with prime ideals!

Example 2.1. Consider $S^3 = \langle x^2 + y^2 - 1 \rangle$ in $\mathbb{C}[x, y]$. Its defining polynomial defines a prime ideal of height one, and therefore represents a 1-cycle in $Z_1(\mathbb{C}[x, y])$. However this cycle vanishes in CH_1.

One usually defines the Chow groups $\mathrm{CH}_*(X)$, and then when X is an n-dimensional variety which is sufficiently nice, we can define $\mathrm{CH}^k(X) := \mathrm{CH}_{n-k}(X)$. That is, we define some cohomologically graded Chow groups by forcing Poincaré duality to hold.

Example 2.2. We have that $\mathrm{CH}^*(\mathbb{P}^n) = \mathbb{Z}[H]/H^{n+1}$, where H is the hyperplane class.

Example 2.3. We have that $\mathrm{CH}^*(\mathbb{A}^n)$ is $\mathbb{Z}$ in degree zero, and 0 elsewhere.

Remark 2.4. The graded set $\mathrm{CH}^*(X)$ has a ring structure given by intersection. In order to show that this works, you need to make sure that you can always find equivalent objects that meet transversely. This uses *Chow's moving lemma*, or perhaps *deformation to the normal cone.*

Remark 2.5. Associated to any *algebraic vector bundle* $E \to X$, we can associate to it *algebraic Chern classes* $c_i(E) \in \mathrm{CH}^i(X)$. These are defined inductively, much like in topology.

Big idea: Suppose X is an affine algebraic variety, and $E \to X$ is an *algebraic* vector bundle. Then its Chern classes in singular cohomology should come in some way from its Chern classes in the Chow ring.

Let's make this a bit more precise. Suppose that X is a sufficiently nice complex variety. Then suppose we have a class in the Chow ring corresponding to a subvariety. By taking complex points, this subvariety becomes a complex submanifold, and therefore represents a singular cohomology class.

In other words, the forgetful functor $\mathtt{Aff}_{\mathbb{C}} \to \mathtt{Top}$ induces a comparison map between cohomology:

$$\mathrm{cl} : \mathrm{CH}^k(X) \to H^{2k}(X; \mathbb{Z}).$$

This is called a *cycle class map.*

Understanding the cycle class map is a massive undertaking. For example, certain nice complex varieties have a cohomology class called the *Hodge class.* Knowing whether this lies in the image of the cycle class map is the *Hodge conjecture*, one of the Millennium problems.

For us, we are interested in which vector bundles admit an algebraic structure. From this discussion we see a necessary condition.

Proposition 2.6. The cycle class map takes algebraic Chern classes to topological Chern classes.

Remark 2.7. If a topological vector bundle is *algebraizable* (meaning that it admits an algebraic structure), then its Chern classes lie in the image of the cycle class map.

Question: Is the converse true?

3. Motivic stuff

Serre's FAC set up a correspondence between topological vector bundles on a manifold, and finitely generated projective modules over its ring of functions. Topologically, we have that the procedure of taking vector bundles are homotopy invariant. That is,

$$\mathtt{Vect}_r^{\mathrm{top}}(X) \cong \mathtt{Vect}_r^{\mathrm{top}}(X \times \mathbb{C}^n).$$

We can ask whether the same is true algebraically.

Conjecture 3.1. (Bass–Quillen Conjecture) Let A be a ring. Then every vector bundle over $A[t_1, \dots, t_n]$ is extended from A. That is, there is a bijection

$$\begin{aligned} \mathtt{Vect}_r^{\mathrm{alg}}(A) &\xrightarrow{\sim} \mathtt{Vect}_r^{\mathrm{alg}}(A[t_1, \dots, t_n]) \\ M &\mapsto M \otimes_A A[t_1, \dots, t_n]. \end{aligned}$$

Theorem 3.2. (Lindel, 1981) The Bass–Quillen theorem holds if A contains a field.

This begs the question: *is there something homotopic happening for algebraic vector bundles?*

How did we look at vector bundles homotopically in the topological setting? We had that

$$\mathtt{Vect}_r^{\mathrm{top}}(X) = [X, \mathrm{Gr}(k, \infty)].$$

Let's think about how we might do this in the world of varieties. Could we attempt to map into an infinite Grassmannian in order to get algebraic vector bundles?

Problem 1: This "infinite Grassmannian" is *not a variety.*

Topologically, we relied on our ability to take colimits of finite Grassmannians in order to build $\mathrm{Gr}(k, \infty)$. In the world of varieties, this won't work — due to the fact that varieties are not a *cocomplete category.* They don't admit all colimits.

They do admit some colimits however! We can build varieties by gluing things together. Consider the following pushout diagram:

$$\begin{array}{ccc} \mathbb{C}-0 & \xrightarrow{z} & \mathbb{C} \\ {\scriptstyle 1/z}\downarrow & \ulcorner & \downarrow \\ \mathbb{C} & \longrightarrow & \mathbb{CP}^1. \end{array}$$

But most colimits don't exist. So we should get a little clever about this. We can "inject" the category of varieties with a homotopy theory by embedding it into a setting where it has a homotopy theory. The "free/universal" way to do this (c.f. Dugger) is to look at functors from varieties to spaces

$$y : \mathrm{Var} \hookrightarrow \mathrm{Fun}(\mathrm{Var}^{\mathrm{op}}, \mathtt{Top}).$$

Problem 2: This embedding doesn't preserve colimits!

So we embedded this into a model category, but we've lost our capacity to recognize how varieties were glued together (i.e. we don't remember the colimits we started with). This is an issue, since this is an essential aspect of the category of varieties. In order to rectify this, we "reintroduce" these diagrams by forcing certain things to be equivalent in this functor category. We will be vague about this, but for the experts, we are localizing at a Grothendieck topology. When we do this, we get something that we might call an *algebraic homotopy category*:

$$\mathcal{H}_{\mathrm{alg}}(\mathbb{C}) = L_\tau \mathrm{Fun}(\mathrm{Var}_{\mathbb{C}}^{\mathrm{op}}, \mathtt{Top}).$$

The upshot is that the infinite Grassmannian lives here *and* we can look at homotopy classes of maps to it!

$$\mathtt{Vect}_r^{\mathrm{alg}}(X) = [X, \mathrm{Gr}(k,\infty)]_{\mathcal{H}_{\mathrm{alg}}(\mathbb{C})}.$$

Problem 3: We still don't have an equivalence

$$[X, \mathrm{Gr}(r,\infty)]_{\mathcal{H}_{\mathrm{alg}}(\mathbb{C})} \cong \left[X \times \mathbb{A}_k^1, \mathrm{Gr}(r,\infty)\right]_{\mathcal{H}_{\mathrm{alg}}(\mathbb{C})}.$$

In order to fix this, we will just force the projection $X \times \mathbb{A}_k^1 \to X$ to be a weak equivalence. After doing this, we get a new category, which we might call the *motivic homotopy category.* Let's denote this by $\mathcal{H}_{\mathrm{mot}}(\mathbb{C})$.

Definition 3.3. We define the set of *motivic vector bundles* by

$$\mathtt{Vect}_r^{\mathrm{mot}} = [X, \mathrm{Gr}(r,\infty)]_{\mathcal{H}_{\mathrm{mot}}(\mathbb{C})}.$$

Huge idea: The forgetful functor from varieties to spaces factors through the motivic category. That is, we have a sequence of maps

$$\mathtt{Vect}_r^{\mathrm{alg}}(X) \to \mathtt{Vect}_r^{\mathrm{mot}}(X) \to \mathtt{Vect}_r^{\mathrm{top}}(X).$$

So in order to check if a topological vector bundle admits an algebraic structure, there is an intermediate step we can ask about, which is whether that topological vector bundle admits a motivic structure!!

Theorem 3.4. (Morel (2012), Schlichting (2015), Asok–Hoyois–Wendt (2015)) If X is smooth and affine, then the map is an isomorphism

$$\mathtt{Vect}_r^{\mathrm{alg}}(X) \xrightarrow{\sim} \mathtt{Vect}_r^{\mathrm{mot}}(X).$$

Corollary 3.5. In order to check if a topological bundle over a smooth algebraic complex manifold is algebraizable, it suffices to check if it admits a motivic structure!

Remark 3.6. In the motivic setting, we can always assume that a smooth variety is affine without loss of generality, since via Jouanalou's trick, there exists a fibration $\widetilde{X} \to X$, so that $\widetilde{X}$ is smooth and affine, and the fibers are affine (in particular, they are contractible).

Example 3.7. (Asok–Fasel–Hopkins, 2018) There exists smooth complex affine varieties of dimension ≥ 4, and plane bundles on them, all of whose Chern classes are algebraizable, that do not admit a motivic structure.

This provided the first counterexample to the algebraization problem!! The construction itself uses Steenrod squaring operations on Chow groups, following work of Voevodsky and Brosnan, and could encapsulate an entire talk of its own.