

# PIMS Workshop on Arithmetic Topology: Mini-Courses

Notes by Thomas Brazelton

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## Abstract

These notes are from the mini-courses at PIMS workshop on arithmetic topology, held at UBC in summer of 2019. Any errors or typos in these notes should be attributed to myself, not the lecturer. Thank you to both the organizers and the lecturers for this conference.

**Disclaimer:** These notes were tex'd live during the lectures and not edited afterwards. There are likely typos and incorrect statements, so please reach out to me with any edits.

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## Opening Remarks

Melanie Wood

**Number Theory** — *arithmetic statistics*: asks questions about number theoretic objects, for example number fields  $K/\mathbb{Q}$  and other objects like elliptic curves  $E/\mathbb{Q}$ . How many are there (in an asymptotic sense) and how many have various properties (e.g. trivial class group, rank 0 elliptic curve)?

**Minicourse:** Wei Ho

The field  $\mathbb{Q}$  is a lot like  $\mathbb{F}_q(t)$ , called *global fields* (these and their finite extensions). Much of number theory can be done uniformly over global fields. The fields  $\mathbb{F}_q(t)$  and its finite extensions are also function fields of geometric objects. We think of  $\mathbb{F}_q(t)$  as functions on  $\mathbb{P}_{\mathbb{F}_q}^1$  and we think of  $K/\mathbb{F}_q(t)$  as functions on some curve  $C$ , where  $C$  is a smooth projective irreducible curve over  $\mathbb{F}_q$ . Thus we can study their algebraic geometry over  $\mathbb{F}_q$ .

A curve  $C \rightarrow \mathbb{P}_{\mathbb{F}_q}^1$  is a lot like  $X \rightarrow \mathbb{P}_{\mathbb{C}}^1$ , i.e. a Riemann surface. Thus algebraic geometry can sometimes relate the study of varieties over  $\mathbb{F}_q$  and over  $\mathbb{C}$  (we'll see that we can relate cohomology, intersection theory, etc. on both sides).

$$\begin{array}{ccc} [\text{Arithmetic Statistics of number fields}] & \xleftarrow{\text{analogy}} & [\text{Arithmetic Statistics over finite fields}] \\ & \xleftarrow{\text{Theorems}} & [\text{Topology of Moduli Spaces}] \end{array}$$

**Minicourses:** Jordan Ellenberg, Benson Farb

**Topology** — *homological stability*: given spaces  $X_1, X_2, \dots$  the question of homological stability is studying  $H_i(X_n; \mathbb{Z})$  (or coefficients in  $\mathbb{Q}$ , etc) as  $n \rightarrow \infty$ . For nice sequences, we could ask:

1. Does  $H_i$  stabilize in its isomorphism class? Does its dimension stabilize?
2. Are there maps  $X_n \rightarrow X_{n+1}$  that induce the stabilization?
3. What does it stabilize to? And if it doesn't stabilize, is there some higher stabilization occurring (e.g. representation stability)?

**Examples.**

- $\text{Conf}^n X$  where  $X = \mathbb{R}^2$  (Arnol'd, 69), and for more general  $X$  (McDuff, Segal 70's, Church '12).
- $M_g$ , the moduli space of elliptic curves (Harer '85)
- $B\text{SL}_n(\mathbb{Z})$  (Borel, '77 with  $\mathbb{Q}$  coefficients)

- $\text{Map}(C, X)$  of increasing degree for a curve  $C$  (Segal, '79)

We will also talk about  $E_2$ -algebras, taking  $X = \coprod_n X_n$  with a multiplication  $X_n \times X_m \rightarrow X_{n+m}$ . The  $E_2$ -algebra is a structure on  $X$  so that one can present  $X$  “as an algebra.” This is a powerful way to study stability in  $n$ .

**Minicourse:** Søren Galatius

Additionally, we will discuss  $\mathbb{A}^1$ -homotopy theory. The model is to do homotopy theory by replacing the interval with  $\mathbb{A}^1$ . This machinery connects the study of any variety  $X/k$  with  $X/\mathbb{C}$ , especially as it relates to intersection theory. This can be used to prove “enriched counts.”

**Minicourse:** Kirsten Wickelgren

Finally, we will study the Grothendieck ring of varieties, which is a place where one can consider the pieces of which algebraic varieties are built. As  $\mathbb{P}^1$  is built of  $\mathbb{A}^1$  and a point, we have that  $[\mathbb{P}^1] = [\mathbb{A}^1] + [\text{pt}]$ . This has connections to the cohomology of the varieties over  $\mathbb{C}$ , and also connections to the cohomology of the varieties over  $\mathbb{F}_q$  (étale cohomology over  $\overline{\mathbb{F}}_q$  with action of Frobenius). This is very related to the count of  $\mathbb{F}_q$ -points of varieties over  $\mathbb{F}_q$ , which is related via the Grothendieck-Lefschetz trace formula.

**Minicourse:** Ravi Vakil.

## Geometric aspects of arithmetic statistics: Part 1

Jordan Ellenberg

**Question 1:** How many squarefree integers are there (in  $[N, 2N]$ )?

This is a question over  $\mathbb{Q}$ . We will replace this by  $\mathbb{F}_q(t)$ . (We could replace this by any number field and any rational function field).

**Question 2:** How many squarefree polynomials in  $\mathbb{F}_q[t]$  are there?

The first problem we encounter is that polynomials don't come in any order, so we don't have an analogy of the interval. We can replace these via

1. How many positive squarefree integers are there with  $|n| \in [N, 2N]$ ?
2. How many monic squarefree polynomials  $f \in \mathbb{F}_q[t]$  exist with  $|f| \in [N, 2N]$ ?

This makes sense for a multiplicative function  $|\cdot| : \mathbb{F}_q[t] \rightarrow \mathbb{Z}$ , but the one we care about is  $|f| = q^{\deg(f)}$ . This valuation only takes values which are powers of  $q$ , so let's think about counting  $f$  with  $|f| = q^n$ ; that is,  $\deg f = n$ .

**Answer 1:**  $\sim \frac{6}{\pi^2} N$ .

**Answer 2:**  $\sim q^n - q^{n-1} = \left(1 - \frac{1}{q}\right) q^n$ .

These are the same answer although they cosmetically don't look the same. In the philosophy that  $N = q^n$ , we have that

$$\frac{6}{\pi^2}N = \zeta(2)^{-1}N$$

$$q^n - q^{n-1} = \zeta_{\mathbb{F}_q[t]}(2)^{-1}N.$$

*Geometric version:* How many monic squarefree polynomials in  $\mathbb{C}[z]$  with degree  $n$ ? This is a space, and we want to study its topology. What is the cohomology of the space of monic squarefree polynomials in  $\mathbb{C}[z]$  of degree  $n$ ?<sup>1</sup>

This is worked out by Arnold, 1969: the space of squarefree monic polynomials of degree  $n$  over  $\mathbb{C}$  is  $\text{Conf}^n \mathbb{C}$ , the space of unordered distinct  $n$ -tuples of elements of  $\mathbb{C}$ . This correspondence is given by:

$$p(t) \mapsto \{\text{roots of } p(t)\}$$

$$\prod_{i=1}^n (t - z_i) \leftrightarrow \{z_1, \dots, z_n\}.$$

It turns out that  $\text{Conf}^n \mathbb{C}$  is a  $K(\pi, 1)$ . Loops in  $\text{Conf}^n \mathbb{C}$  turn out to be braids, so  $\pi$  is the *Artin braid group*  $\text{Br}_n$  for braids on  $n$  strands. So we have that

$$H^i(\text{Conf}^n \mathbb{C}, \mathbb{Q}) = H^i(\text{Br}_n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & i = 0 \\ \mathbb{Q} & i = 1, n > 0 \\ 0 & i > 1. \end{cases}$$

That is,  $\text{Br}_1 = \{e\}$ , which has the correct cohomology above. The group  $\text{Br}_2 = \mathbb{Z}$ , since two braids can only wrap around each other, which has the cohomology of a circle. Thus the above computation of Arnold tells us that rational cohomology of  $\text{Br}_n$  stays a circle  $S^1$ .

We claim that this is the same as answers 1 and 2. Why is this the same?

**Grothendieck Lefschetz Trace Formula:** If  $X$  is a smooth variety over  $\mathbb{F}_q$ , then

$$|X(\mathbb{F}_q)| = \sum (-1)^i \text{Tr}(\text{Frob}_q | H_{\text{ét},c}^i(X/\overline{\mathbb{F}}_q, \mathbb{Q}_\ell)),$$

the trace of the action on the operator on the compactly supported étale cohomology. Under favorable circumstances, i.e. when  $X$  is a variety over  $\mathbb{Z}$  (or some extension of  $\mathbb{Z}$ ), then

$$H^i(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H_{\text{ét}}^i(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_\ell).$$

Frobenius is the map which sends  $x \mapsto x^q$ , thus the fixed points are  $X(\mathbb{F}_q)$ , the  $\mathbb{F}_q$ -rational points of  $X$ . Now take  $X = \text{Conf}^n$ , the moduli space of monic squarefree

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<sup>1</sup>So we are taking “cardinality” and “set” and swapping them out for “cohomology” and “space.”

polynomials of degree  $n$ . Scheme-theoretically, we think about this as the scheme giving monic squarefree polynomials of degree  $n$  for any field. That is, its  $\mathbb{F}_q$  points are

$$\text{Conf}^n(\mathbb{F}_q) = \{\text{monic squarefree degree } n \text{ polynomials in } \mathbb{F}_q[t]\}.$$

Thus we get the following formula:

$$\begin{aligned} |\text{Conf}^n(\mathbb{F}_q)| &= \sum_{i=0}^1 (-1)^i \text{Tr}(\text{Frob} | H_{\acute{e}t,c}^i(\text{Conf}^n(\mathbb{F}_q), \mathbb{Q}_\ell)) \\ &= \text{Tr}(\text{Frob} | H^0(\text{Conf}_{\mathbb{F}_q}^n, \mathbb{Q}_\ell)) - \text{Tr}(\text{Frob} | H^1(\text{Conf}_{\mathbb{F}_q}^n, \mathbb{Q}_\ell)). \end{aligned}$$

The action of Frob on  $H^0$  is  $q^n$  and the action on  $H^1$  is  $q^{n-1}$ . In fact, Answer 2 is equal to  $q^n - q^{n-1}$  unless  $n = 1$ , in which case it is just  $q$ .

*Overall philosophy:*

- analogies between counting over  $\mathbb{Z}$  and counting over  $\mathbb{F}_q[t]$
- topology over  $\mathbb{C}[z]$  can sometimes be used to prove things about counting over  $\mathbb{F}_q[t]$ .

What if we didn't know the cohomology of the braid group  $H^i(\text{Conf}^n, \mathbb{C})$ ? We would still know that  $H^0(\text{Conf}^n \mathbb{C}, \mathbb{Q}) = \mathbb{Q}$  since the space is connected for all  $n$ .

As  $q \rightarrow \infty$ , everything except  $H^0$  becomes negligible (compared to the contribution of  $H^0$ ). Thus as  $q \rightarrow \infty$ , the probability that a given polynomial is squarefree becomes:

$$\lim_{q \rightarrow \infty} \frac{\#\{\text{monic sqfree polys of degree } n\}}{\#\{\text{monic polys of degree } n\}} = 1.$$

**Question:** What happened if we chose something else that  $\mathbb{F}_q[t] \subseteq \mathbb{F}_q(t)$ ?

**Answer:** If we take a Riemann surface and chop out some identified point at  $\infty$ , then we could take the configuration space of points with the condition that they can't be infinity. That is, we could think of the configuration space as

$$\begin{aligned} \text{Conf}^n \mathbb{C} &= \text{Conf}^n \mathbb{A}^1(\mathbb{C}) \\ &= \text{Conf}^n(\mathbb{P}^1(\mathbb{C}) - \{\infty\}). \end{aligned}$$

**Question:** For a topologist, all pts on Riemann surface are the same, but algebraically they could be different.

**Answer:** On  $\mathbb{P}^1$  they really are the same. Over a larger curve, the topology will be the same, but the action of Frobenius on the space may depend upon the point which was deleted. In these situations, we have to determine what the action is, or we might have to bound it.

**Question:** Why do we only delete one point?

**Answer:** Exercise for the grad students: what if I deleted two points? What is  $\text{Conf}^n \mathbb{G}_m$  a moduli space of?

**Question:** In the function field case, we get to choose which point is the point at infinity. Over  $\mathbb{Q}$ , is there freedom to pick a prime at infinity? Is there some non-archimedean place? Could we set it up so that the  $p$ -adic valuation plays the role the classical valuation plays?

**Answer:** We get powers of  $p$  in the denominator,

$$\mathbb{Z} = \{x \in \mathbb{Q} : |x|_v \leq 1 \forall v \neq \infty\}.$$

We could make this something like

$$\{x \in \mathbb{Q} : |x|_v \leq 1 \forall v \neq 17\},$$

which is not closed anymore.

## Geometric aspects of arithmetic statistics: Part 2

Jordan Ellenberg

Two main parts:

1. Cohen-Lenstra heuristics
2. statistics of factorizations of polynomials.

**Question:** Let  $\ell$  be an odd prime. What is the average value of  $|\text{Cl}_{\mathbb{Q}(\sqrt{-d})}[\ell]|$  for squarefree  $d \in [N, 2N]$ ?

Cohen-Lenstra heuristics tell us that this average should be 2. What is the  $\mathbb{F}_q(t)$ -analog? Instead of  $\mathbb{Q}(\sqrt{-d})$  for squarefree  $d \in [N, 2N]$ , we take  $N = q^n$  and then look at  $\mathbb{F}_q(t)(\sqrt{f})$  for  $f$  a monic squarefree polynomial of degree  $n$ . This is exactly  $\mathbb{F}_q(C)$  for  $C$  a hyperelliptic curve of genus  $\frac{n-1}{2}$ . The analog of the class group is the  $\mathbb{F}_q$ -points of its Jacobian variety  $\text{Jac}(C)(\mathbb{F}_q)$ , and so we are now asking: what is the average size of  $\text{Jac}(C)(\mathbb{F}_q)[\ell]$  as  $C$  ranges over hyperelliptic curves  $C = C_f : y^2 = f(x)$ , where  $f$  is monic squarefree of degree  $n$ .

What is the moduli space in this setting? We can define  $\text{Conf}^n(\ell)$  to be the moduli space of pairs  $(f, P)$  where  $f$  is monic and squarefree of degree  $n$ , and  $P$  is a point of order  $\ell$  on  $\text{Jac}(C_f)$ .<sup>2</sup> We would call this a “moduli space of hyperelliptic Jacobians with  $\ell$ -level structure.” We claim counting points on this space is tantamount to the question above.

**Question:** What is the expected value, for a random  $C$ :

$$E_C [\#\{P \in \text{Jac}(C)(\mathbb{F}_q)[\ell]\}] = \frac{\sum_f \#\{(f, P)\}}{\sum_f 1} = \frac{|\text{Conf}^n(\ell)(\mathbb{F}_q)|}{|\text{Conf}^n(\mathbb{F}_q)|}.$$

<sup>2</sup>This looks like the moduli space of elliptic curves with level structure.

We hope that  $\lim_{n \rightarrow \infty} \frac{|\text{Conf}^n(\ell)(\mathbb{F}_q)|}{|\text{Conf}^n(\mathbb{F}_q)|} = 1$  (since we are taking out the point 0, as we are counting points of exact order  $\ell$ ). We have that  $\text{Conf}^n(\ell)$ , like  $\text{Conf}^n$ , is a scheme over  $\text{Spec } \mathbb{Z}$ . What are its complex points? We note that there is always a map  $\text{Conf}^n(\ell) \rightarrow \text{Conf}^n$  given by forgetting the point  $P$ , which is finite of degree  $\ell^{2g} - 1 = \ell^{2 \deg(C)} - 1$ .

So  $\text{Conf}^n(\ell)(\mathbb{C}) \rightarrow \text{Conf}^n(\mathbb{C}) = K(\pi, 1)$  corresponds to the action of  $\pi_1(\text{Conf}^n) = \text{Br}_n$  on some set of size  $\ell^{2g} - 1$ . We recall that the braid group has a representation to  $\text{Sp}_{2g}(\mathbb{Z})$  which is the action of the double cover branched at the  $2n$  points (this is a specialization of a Burau representation to  $t = -1$ ). And then  $\pi_1(\text{Conf}^n(\ell))$  is a finite-index subgroup, which is the stabilizer in this action of a nonzero point of  $(\mathbb{Z}/\ell\mathbb{Z})^{2g}$ . This is known as a “congruence subgroup” of the braid group.

What can we say about the cohomology of  $\text{Conf}^n(\ell)$ ? For instance,  $H^0(\text{Conf}^n(\ell)) = \# \text{orbits of the braid group } \text{Br}_n \text{ acting on } (\mathbb{Z}/\ell\mathbb{Z})^{2g} - \{0\}$ , which is exactly 1 since  $\text{Br}_n \rightarrow \text{Sp}_{2g}(\mathbb{Z}/\ell\mathbb{Z})$  when  $\ell$  is an odd prime<sup>3</sup> (A Compu, J-K Yu, Achter-Pries, Holl).

Thus

$$\lim_{n \rightarrow \infty} \lim_{q \rightarrow \infty} \frac{|\text{Conf}^n(\ell)(\mathbb{F}_q)|}{|\text{Conf}^n(\mathbb{F}_q)|} = 1.$$

**Theorem** (Ellenberg, Venkatesh, Westerland).

- (homological stability) there exists  $\alpha > 0$  (depending on  $\ell$ ) so that, for all  $i < \alpha n$ , we have

$$H_{\acute{e}t}^i(\text{Conf}^n(\ell), \mathbb{Q}_{\ell'}) \cong H_{\acute{e}t}^{i+1}(\text{Conf}^{n+1}(\ell), \mathbb{Q}_{\ell'}).$$

- (Betti bound) we have that  $\dim H_{\acute{e}t}^i(\text{Conf}^n(\ell), \mathbb{Q}_{\ell'}) < C^i$

Given these two facts, they imply the following statement towards Cohen-Lenstra:

$$1 - g(q) < \liminf \frac{|\text{Conf}^n(\ell)(\mathbb{F}_q)|}{|\text{Conf}^n(\mathbb{F}_q)|}, \quad \limsup \frac{|\text{Conf}^n(\ell)(\mathbb{F}_q)|}{|\text{Conf}^n(\mathbb{F}_q)|} < 1 + f(q)$$

for  $f(q), g(q) \rightarrow 0$  as  $q \rightarrow \infty$ .

**Further notions:** What is the average number of linear factors of a squarefree polynomial? One way to set this up: define  $LF(f)$  to be the number of linear factors, which is a  $\mathbb{Z}$ -valued function on  $\text{Conf}^n(\mathbb{F}_q)$ . Grothendieck-Lefschetz is more general than counting points on a space  $X$ ; if  $\mathcal{F}$  is a local system on  $X$ , we can relate the cohomology of  $\mathcal{F}$  to the stalks of  $\mathcal{F}$ .

In the above example, when  $X = \text{Conf}^n$ , we should think about  $\mathcal{F}$  as a representation of the braid group  $\text{Br}_n$ , so let  $\mathcal{F}$  be  $V_n$ , the permutation representation on the strands. So Grothendieck-Lefschetz gives

$$\sum (-1)^i \text{Tr}(\text{Frob} | H_c^i(X, \mathcal{F})) = \sum_{x \in X(\mathbb{F}_q)} \text{Tr}(\text{Frob} | \mathcal{F}_x).$$

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<sup>3</sup>This is a “big monodromy statement” which means image of  $\pi_1$  is big, so the action is transitive, so the number of orbits is small.

For the permutation representation  $V$ , the action of Frobenius on  $\mathcal{F}_x = \mathcal{F}_f$  (where  $f$  is squarefree polynomial of dimension  $n$ ) is just the permutation action on the roots of  $f$ . Then the trace of a permutation representation is the number of one's on the diagonal— in this setting,  $\text{Tr}(\text{Frob}|\mathcal{F}_f)$  is the number of roots fixed by Frobenius, i.e. the number of linear factors:

$$\text{Tr}(\text{Frob}|\mathcal{F}_f) = LF(f).$$

Thus computing  $\sum_{f \in \text{Conf}^n(\mathbb{F}_q)} LF(f)$  amounts to computing  $H^i(\text{Conf}^n, V_n)$ . We should anticipate some kind of homological stability in this latter term. Since  $V$  factors through the map  $\text{Br}_n \rightarrow S_n$ , we can take larger coefficients (say, the group ring of  $S_n$ ) and then tensor down to get, by Shapiro's Lemma:

$$H^i(\text{Conf}^n(\mathbb{F}_q), \mathbb{C}[S_n]) \otimes_{\mathbb{C}[S_n]} V_n = H^i(P\text{Conf}^n, \mathbb{C}) \otimes_{\mathbb{C}[S_n]} V_n.$$

The representation stability of  $H^i(\text{Conf}^n(\mathbb{F}_q), \mathbb{C}[S_n])$  will be discussed more by Benson Fab.

**Question:** What error terms do you get with this method?

**Answer:** Any time we have a range of stabilization for cohomology which is linear in degree, we get a power-saving error term. This doesn't give better error terms than analytic number theory.

**Question:** Can we work backwards from better error terms in analytic number theory? Should it infer something about cohomology?

**Answer:** Usually not — for the variety  $\mathbb{P}^1 - \{0, \infty\}$  over  $\mathbb{F}_q$  we get  $q - 1$  points. We can't, however, prove that the cohomology is concentrated in degree 0.

## $\mathbb{A}^1$ -enumerative geometry: Part 1

Kirsten Wickelgren

Original work joint with Jesse Kass.

### Intro to $\mathbb{A}^1$ -homotopy theory via $\mathbb{A}^1$ -degree

We can consider a sphere as

$$S^n = \left\{ (x_0, \dots, x_n) \in \mathbb{R}^{n+1} : \sum x_i^2 = 1 \right\} = \mathbb{P}^n(\mathbb{R})/\mathbb{P}^{n-1}(\mathbb{R}).$$

There is a degree map  $\text{deg} : [S^n, S^n] \rightarrow \mathbb{Z}$ . If we have  $f : S^n \rightarrow S^n$ , then  $\text{deg } f$  can be computed as a sum indexed by preimages over a point

$$\text{deg } f = \sum_{q \in f^{-1}(p)} \text{deg}_q f,$$



where we have chosen  $p$  so that it has finitely many preimages  $f^{-1}(p) = \{q_1, \dots, q_N\}$ . The *local degree* is defined as follows: let  $V \ni p$  and  $U \ni q_i$  be small balls around these points so that  $f^{-1}(p) \cap U = \{q_i\}$ . Then we have induced maps

$$S^n \simeq U/\partial U \simeq U/(U - \{q_i\}) \xrightarrow{\bar{f}} V/(V - \{p\}) \simeq S^n.$$

**Note.** We have to be careful about the orientation when we identify  $S^n \simeq U/\partial U$ .

Then we define  $\deg_{q_i} f = \deg \bar{f} \in \mathbb{Z}$ .

We additionally have a formula from differential topology: let  $y_1, \dots, y_n$  be coordinates around  $p$  and  $x_1, \dots, x_n$  be coordinates around  $q_i$ , which both respect an orientation. Then we can represent  $f = (f_1, \dots, f_n) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . Let  $Jf = \left( \frac{\partial f_i}{\partial x_j} \right)$ , and we can write

$$\deg_{q_i} f = \begin{cases} +1 & Jf(q_i) > 0 \\ -1 & Jf(q_i) < 0. \end{cases}$$

We have that  $f$  is an algebraic function over  $\mathbb{C}$ , so degree counts preimages.

By Lannes/Morel, let  $k$  be a field, and consider  $f : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$ . This is a map of spheres in  $\mathbb{A}^1$ -topology. Then we can consider a degree  $\deg f \in \text{GW}(k)$  valued in the *Grothendieck-Witt ring*, which is the group completion of isomorphism classes of non-degenerate symmetric bilinear forms  $\beta : V \times V \rightarrow k$  for  $V \in \text{Vect}_k$  finite-dimensional.

The Grothendieck-Witt ring has a presentation as follows: it is generated by

$$\begin{aligned} \langle a \rangle : k \times k &\rightarrow k \\ (x, y) &\mapsto axy, \end{aligned}$$

for  $a \in (k^\times)/(k^\times)^2$ . This is subject to the following relations

1.  $\langle a \rangle = \langle ab^2 \rangle$
2.  $\langle a \rangle \langle b \rangle = \langle ab \rangle$
3.  $\langle a \rangle + \langle b \rangle = \langle a + b \rangle + \langle ab(a + b) \rangle$  if  $a + b \neq 0$
4.  $\langle a \rangle + \langle -a \rangle = \langle 1 \rangle + \langle -1 \rangle = \mathbb{H}$ , called the hyperbolic form.

**Example.** Over  $\mathbb{C}$ , every two nonzero complex numbers differ by a square, so we get an isomorphism

$$\begin{aligned} \text{rank} : \text{GW}(\mathbb{C}) &\rightarrow \mathbb{Z} \\ \beta &\mapsto \dim V. \end{aligned}$$

**Example.** By Sylvester's formula, we can diagonalize any form, so we get a map

$$(\text{signature}, \text{rank}) : \text{GW}(\mathbb{R}) \rightarrow \mathbb{Z} \times \mathbb{Z},$$

and indeed  $\text{GW}(\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$  is isomorphic to a subgroup.

**Example.** By taking discriminant and rank, we get

$$(\text{signature}, \text{rank}) : \text{GW}(\mathbb{F}_q) \xrightarrow{\cong} \mathbb{F}_q^\times / (\mathbb{F}_q^\times)^2 \times \mathbb{Z}.$$

These invariants show up in  $\mathbb{A}^1$ -homotopy theory. Consider the kernel of the rank map:

$$0 \rightarrow I \rightarrow \text{GW}(k) \xrightarrow{\text{rank}} \mathbb{Z} \rightarrow 0.$$

We call  $I$  the *fundamental ideal*. Then we have a filtration

$$I^n / I^{n+1} \cong K_n^M(k) \otimes \mathbb{Z}/2 \cong H_{\text{ét}}^n(k, \mathbb{Z}/2).$$

This is the Milnor conjecture, proven by Voevodsky. This gives us a sequence of invariants on quadratic forms: rank, discriminant, Hasse-Witt invariant, Aaronson invariant, etc.

Back to Lannes/Morel: we define the degree of  $f : \mathbb{P}_k^1 \rightarrow \mathbb{P}_k^1$  as

$$\deg f = \sum_{q \in f^{-1}(p)} \langle Jf(q) \rangle.$$

If  $q$  is not  $k$ -rational, then  $\langle Jf(q) \rangle$  may lie in  $\text{GW}(k(q))$ . Thus we apply a transfer map — if  $k \subseteq L$  is a finite separable extension, then we have a trace

$$\begin{aligned} \text{Tr}_{L/k} : \text{GW}(L) &\rightarrow \text{GW}(k) \\ \beta &\mapsto (V \times V \xrightarrow{\beta} L \xrightarrow{\text{Tr}_{L/k}} k), \end{aligned}$$

and since the extension is separable, the form remains non-degenerate.

So in defining the global degree, we pick  $p$  so that  $f^{-1}(p)$  is finite and  $Jf(q) \neq 0$ . We note that the global degree does not depend on  $p$ .

**Examples.**

1.  $\deg(z \mapsto az) = \langle a \rangle$
2.  $\deg(z \mapsto z^2) = \langle 1 \rangle + \langle -1 \rangle$ .

*Theme:* We'll use the degree to count algebro-geometric objects, where the count will be valued in  $\text{GW}(k)$ . This will record arithmetic and geometric properties of the objects. Applying invariants, we get more traditional counts.

**Example.** Let  $X \subseteq \mathbb{P}_k^3$  be a smooth cubic surface. Then  $X_{\bar{k}}$  has 27 lines. A line  $L$  on  $X$  is defined over  $k(L)$ .

**Theorem** (Kass-Wickelgren). When  $\text{char}(k) \neq 2$ . Then

$$\sum_{\text{lines } L \text{ on } X} \text{Tr}_{k(L)/k} \text{Type}(L) = 15 \langle 1 \rangle + 12 \langle -1 \rangle.$$

As we travel along a line, paying attention to the orientation of its tangent space, we can see whether the plane rotates around the line  $L$ .

We define  $\text{Type}(L)$  via a map  $p \mapsto T_p X$  for  $p \in L$ . This map has two pause points (where the tangent plane momentarily stops rotating) over  $L(\bar{k})$ . These are defined over some quadratic extension  $k(L)[\sqrt{D}]$ . Then the type of  $L$  is  $\text{Type}(L) = \langle D \rangle \in \text{GW}(k(L))$ .

Sabrina Pauli will discuss results relating to lines on a quintic 3-fold.

Morel defined  $\text{deg} : [\mathbb{P}^n / \mathbb{P}^{n-1}, \mathbb{P}^n / \mathbb{P}^{n-1}]_{\mathbb{A}^1} \rightarrow \text{GW}(k)$ . To make sense of this, we need  $\mathbb{P}^n / \mathbb{P}^{n-1}$  as an object somewhere, since it is not a scheme, and we need to understand  $\mathbb{A}^1$ -homotopy classes of maps  $[-, -]_{\mathbb{A}^1}$ .

We can view  $\mathbb{P}^n / \mathbb{P}^{n-1}$  as the colimit

$$\text{colim}(* \leftarrow \mathbb{P}^n \rightarrow \mathbb{P}^{n-1}).$$

We will have a notion of weak equivalence, and as a result we will want to replace colimits by homotopy colimits. We can do this via a simplicial model category or an  $\infty$ -category. These have a notion of weak equivalence and have an associated homotopy category. Taking  $s\text{Pre}(Sm_k)$  allows us to freely adjoin colimits. We can think of this as containing  $Sm_k$  via the Yoneda embedding

$$\begin{aligned} y : Sm_k &\rightarrow s\text{Set}^{Sm_k^{\text{op}}} \\ X &\mapsto \text{Map}(-, X). \end{aligned}$$

Similarly, for any  $T \in s\text{Set}$  we can view it in the category  $s\text{Pre}(Sm_k)$  as the constant sheaf at  $T$ .

Another example of a colimit is given as follows: if  $U, V \subseteq X$  are open subschemes of a common scheme, then we can build their union as a colimit

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & \lrcorner & \downarrow \\ V & \longrightarrow & U \cup V \end{array}$$

After freely adding colimits, we want the colimits we had in  $Sm_k$  like the one above to remain colimits. This is done via Bousfield localization, which will force the open covers in a given topology to give colimits. We force  $\text{cosk}^0 \coprod U_\alpha \xrightarrow{\sim} X$  to be a weak equivalence, for  $\coprod U_\alpha$  an open cover of  $X$ . Thus we get a category of sheaves, and finally we perform one last Bousfield localization  $L_{\mathbb{A}^1}$  to force maps of the form  $X \times \mathbb{A}^1 \rightarrow X$  to be weak equivalences.

Finally, we obtain a category  $\text{Spc}$  which we call the *unstable motivic homotopy category*.

**Remark.** We can define a local degree  $\text{deg}_q f \in \text{GW}(k)$  because for  $q$  a smooth point of  $X$ , then  $U/(U - \{q\}) \simeq \mathbb{P}^n / \mathbb{P}^{n-1} \wedge \text{Spec } k(q)_+$ . Morel then constructs, for  $k$  a field, the degree map to  $\pi_0(\xi_k) \cong \text{GW}(k)$ , which he shows is an isomorphism for  $n \geq 2$ .

**Remark.** For  $Z \subseteq X$  of codimension  $i$ , and irreducible, we can form  $\widetilde{\text{CH}}^i(X)$  oriented Chow, where cycles are formal sums  $(Z, \beta \in \text{GW}(k))$ . This gives us intersection theory.

Next up: degrees between maps of smooth schemes for counting rational curves.

## Point counting in topology: Part 1

Benson Farb

Let  $X$  be a variety over  $\mathbb{Z}$ .

*Theme:* Understand the relationship between  $X(\mathbb{C})$  and arithmetic statistics of  $X(\mathbb{F}_q)$ .

We will talk about smooth hypersurfaces in  $\mathbb{P}^n$ . This will be today's main example. Let  $F \in \mathbb{C}[x_0, \dots, x_n]_{(d)}$  a degree  $d$  homogeneous polynomial, and let  $Z_F \subseteq \mathbb{P}^n$  be the associated hypersurface. We define

$$\begin{aligned} U_{d,n} &:= \{\text{smooth, degree } d \text{ hypersurfaces } X \subseteq \mathbb{P}^n\} \\ &= \mathbb{P}^{\binom{d+n}{n}-1} - \Sigma_{d,n}, \end{aligned}$$

where we are subtracting out the singular ones. To be singular means there exists a point where  $F$  vanishes and all the partials with respect to  $x_i$ 's are zero. This is a resultant of polynomials.

Everything we say today is related to  $U_{d,n}/\text{PGL}_{n+1}(\mathbb{C})$ , but for simplicity we will ignore that.

**Basic questions:**

Understand the topology (e.g. co-homology) of	Arithmetic statistics
Single variety $Z_F(\mathbb{C})$	$ Z_F(\mathbb{F}_q)  = ?$ what is the expected value? What are the higher moments? etc.
Moduli space $U_{d,n}(\mathbb{C})$	count $ U_{d,n}(\mathbb{F}_q)  = ?$
Varieties with extra data	count these over $\mathbb{F}_q$ .

For example, we have the “universal degree  $d$  hypersurface in  $\mathbb{P}^n$ ,” denoted  $E_{d,n} := \{(Z_F, P) : F(P) = 0\}$  where  $P$  is a point on  $Z_F$ , we get  $E_{d,n} \rightarrow U_{d,n}$ . We see that  $U_{d,n}$  is connected, and since  $E_{d,n}$  is a fiber bundle, all the fibers are diffeomorphic. So it only depends upon the degree of the polynomial.

**A paradigm: smooth cubic surfaces:** let  $U_{3,3} = \mathbb{P}^{19} \setminus \Sigma_{3,3}$ .

**Theorem** (Cayley-Salmon, 1849). Every smooth cubic surface  $X \subseteq \mathbb{P}^3$  contains precisely 27 lines.

This is the first instance where we can use topology to show the existence of solutions without actually solving them.

*Proof.* Form  $U_{3,3}(L) = \{(S, L) : L \subseteq S \text{ is a line}\} \subseteq U_{3,3} \times \text{Gr}(1, 3)$ . We take a projection

$$\begin{aligned} \pi : U_{3,3}(L) &\rightarrow U_{3,3} \\ (S, L) &\mapsto S. \end{aligned}$$

This is a covering space. Since  $U_{3,3}$  is connected, we have that  $\pi^{-1}(x)$  does not depend on the choice of  $x$ . That is, all smooth cubic surfaces have the same number of lines. Finally, we can just look at the Fermat cubic  $x^3 + y^3 + z^3 + w^3 = 0$  which has 27 lines, e.g.

$$\{x = -y\} \cap \{z = -w\}.$$

Since we can permute the variables and multiply each one by a cube root of unity, we get 27 solutions. □

Since we have a 27-sheeted cover, its fiber gives a representation from a loop in the base to the fiber of a chosen basepoint, i.e.

$$\rho : \pi_1(U_{3,3}) \rightarrow \text{Perm}(\pi^{-1}(x_0)) \cong S_{27}.$$

**Theorem** (Jordan). We have that  $\text{im}(\rho) \cong W(E_6)^4$ , and is also isomorphic to an automorphism group of the intersection graph of the 17 lines, which has a line between  $L_i$  and  $L_j$  iff  $L_i \cap L_j = \emptyset$ .

We have that  $|W(E_6)| = 51,840$ .

We can see a sequence of covers

$$\begin{array}{ccc} U_{3,3}(27) & \longrightarrow & U_{3,3}(27)/(S_6 \times \mathbb{Z}/2) \cong \{(S, \text{double six})\} \\ \downarrow & & \downarrow \\ & & \vdots \\ \downarrow & & \downarrow \\ W(E_6) \curvearrowright U_{3,3}(L_1, L_2) & & \\ \downarrow & & \downarrow \\ U_{3,3}(L) & & \\ \downarrow & & \downarrow \\ U_{3,3} & & \end{array}$$

**Open Problem 1:** Compute  $H^*(-, R)$  for all these spaces. For  $R = \mathbb{Q}$ , all these computations follow from  $H^*(U_{3,3}; \mathbb{Q})$  with an action of  $W(E_6)$ . For example, by transfer, we have that

$$H^*(U_{3,3}; \mathbb{Q}) \cong H^*(U_{3,3}(27); \mathbb{Q})^{W(E_6)}.$$

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<sup>4</sup>The Weil group of type  $E_6$ .

So we just have to figure out  $\text{Res}_G^{W(E_6)}$ .

**Theorem** (2019, Vassiliev, Das, Das-O'Connor, Bergvall-Gonnellas).

$$H^*(U_{3,3}(27); \mathbb{Q}) \cong \oplus \text{specific } W(E_6) \text{ irreps.}$$

For example,  $H^2(U_{3,3}(27); \mathbb{Q})$  is the unique 81-dimensional irreducible representation of  $W(E_6)$ .

**Open Problem 2:** Compute  $H^*$  of families of smooth cubic surfaces. E.g. the universal family.

**Theorem** (Das, 2018). Let  $X \rightarrow E_{3,3} \rightarrow U_{3,3}$  be the universal family. Then  $H^*(E_{3,3}; \mathbb{Q}) = \frac{\mathbb{Q}[\alpha_3, \alpha_5, \alpha_7, \eta]}{(\alpha_3^2, \alpha_5^2, \alpha_7^2, \eta^3)}$ .

Method: Vassiliev spectral sequence. By Alexander duality  $\mathbb{P}^n \rightarrow H^*(\Sigma_{3,3})$ , we can look at a poset of strata (of all the ways that smooth cubics can be stratified). Then we consider the associated spectral sequence.

For  $U_{3,3}(L)$  it already becomes much more complicated.

### The arithmetic story

Let  $X/\mathbb{F}_q$ , then  $\text{Frob}_q$  acts on  $X(\overline{\mathbb{F}}_q)$ . Let  $X(\mathbb{F}_q) = \text{Fix}(\text{Frob}_q)$ , then we have that étale cohomology  $H_{\text{ét},c}^*(X/\overline{\mathbb{F}}_q; \mathbb{Q}_\ell)$  comes equipped with an action of Frobenius. We have the G-L trace formula, which says

$$|X(\mathbb{F}_q)| = \sum_{i=0}^{2 \dim_{\mathbb{R}} X} (-1)^i T_i,$$

where  $T_i = \text{Tr}(\text{Frob}_q | H_{\text{ét},c}^i)$  is the trace of the action of Frobenius on étale cohomology.

How to compute the right hand side? We need the number of eigenvalues, as well as their size. Under good circumstances, comparison and basechange tell us that

$$H_{\text{ét},c}^i(X/\overline{\mathbb{F}}_q; \mathbb{Q}_\ell) = H^i(X(\mathbb{C}); \mathbb{Q}) \otimes \mathbb{Q}_\ell.$$

**Theorem** (Deligne). <sup>5</sup> If  $\lambda$  is any eigenvalue of  $\text{Frob}_q$  on  $H_{\text{ét},c}^i$  we have that

1.  $\lambda$  is an algebraic integer
2.  $|\lambda| \leq q^{1/2}$
3.  $|\lambda| = q^{i/2}$  when  $X$  is smooth and projective (Riemann hypothesis over finite fields)
4.  $\text{Tr}_i = \begin{cases} 1 & i = 0 \\ q^d & i = 2d. \end{cases}$

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<sup>5</sup>Resolution of Weil conjectures, also Riemann hypothesis over finite fields.

Given a cycle (cohom class) in a variety, and its dual is represented by algebraic cycles, we get an actual number

**Corollary.** Suppose  $\dim X = d$  and the number of components of  $X$  is  $r$ . Then

$$\limsup_{q \rightarrow \infty} \frac{|X(\mathbb{F}_q)|}{q^d} = r.$$

*Proof.* We have that  $|X(\mathbb{F}_q)| = rq^d + \sum_{i=0}^{2 \dim X - 1} (-1)^i T_i$ , where each  $T_i$  has absolute value  $\leq q^{1/2} b_i$ , where  $b_i$  is the  $i$ th Betti number.  $\square$

**Corollary.** If  $X$  is a smooth projective surface for which  $\pi_1 X = 0$  (e.g. degree  $d \geq 3$  smooth hypersurface in  $\mathbb{P}^3$ ), then

$$|X(\mathbb{F}_q)| = q^2 + 1 + \text{Tr}_2,$$

where all eigenvalues have  $|\lambda| = q$ .

Read Andre Weil's 1954 ICM Talk.

**Theorem** (Manin, 86). If  $X$  is a smooth cubic surface. Then

$$|X(\mathbb{F}_q)| = q^2 + (1 + a)q + 1,$$

where  $a \in \{-3, -2, -1, 0, 1, 2, 3, 4, 6\}$ .

*Proof.* We have the cycle class map, i.e. the  $\mathbb{Z}$ -span of the set of algebraic cycles in  $H^2$  tensored over  $\mathbb{Q}_\ell(-1)$  mapping isomorphically to  $H_{\acute{e}t,c}^2$ .  $\square$

The 27 lines are algebraic cycles, so Frobenius has to permute stuff around. Then the automorphism group of the lines is  $W(E_6)$ . If we fix a line, then Frobenius is acting on  $\mathbb{P}^1$ , and the eigenvalue is just  $q$ . Thus

$$\text{Tr}_2 = q \cdot \chi_{H^2},$$

where the character on  $H^2$  is  $\chi_{H^2} = \chi_{V_{st}d \oplus V_{triv}} = (a + 1)$ . Looking at the character table of  $W(E_6)$ , the characters can only take the values in the set above.

Work of Swinnerton-Dyer shows that all these values can exist. So we can ask how many exist for each  $d$ .

**Theorem** (Das, 2015). 1. The expected number of  $X(\mathbb{F}_q)$  is  $q^2 + q + 1$

2. We get the exact distribution of the eigenvalues of Frobenius. As  $q \rightarrow \infty$ , the distribution is exactly the conjugacy class of  $W(E_6)$ .

This corresponds to the problem of the cohomology  $H^*(E_{3,3})$ . That is,  $\frac{|E_{3,3}(\mathbb{F}_q)|}{|U_{3,3}(\mathbb{F}_q)|}$ .

Other counts (e.g. about lines):

**Theorem** (Das, 2018). The expected number of lines on a smooth cubic surface over  $\mathbb{F}_q$  is 1.

In general: Tomassi, Peters-Steenbrink computed homological stability for  $H^i(U_{d,n}(\mathbb{C}); \mathbb{Q})$  as  $d \rightarrow \infty$ .

**Question:** Why  $E_6$ ?

**Answer:** There is a paper of Manivel that explains this.

## $E_2$ -algebras in spaces: Part 1

Søren Galatius

Joint with Kupers, Randal-Williams.

Let  $\mathcal{D}$  be an operad equivalent to the little disks operad. That is, a sequence of spaces  $\mathcal{D}(n)$  equipped with an action of the symmetric group  $S_n$  and some other conditions, where  $\mathcal{D}(n) \simeq \text{Conf}_n(\mathbb{C})$ . Then we can think of this as a functor

$$\mathcal{D}(X) = \coprod_{n \geq 0} (\mathcal{D}(n) \times X^n) / S_n,$$

We may think of  $\mathcal{D}(n)$  as a space of  $n$ -ary operations  $X^n \rightarrow X$ .

An  $E_2$ -algebra is  $(X, \mu)$  with a structure map  $\mu : \mathcal{D}(X) \rightarrow X$  satisfying some properties. We say a non-unital  $E_2$  algebra is one that satisfies  $\mathcal{D}(0) = \emptyset$ .

An  $E_1$ -algebra is similar but with  $\text{Conf}_n(\mathbb{R})$ .

We have a similar notion in based spaces by taking  $\wedge(\mathcal{D}(n)_+ \wedge X^{\wedge n}) / S^n$ , and on chain complexes by  $C_*(\mathcal{D}) \otimes C^{\otimes n} / S^n$ .

Types of examples:

- found in nature
- ones that you build yourself.

For  $E_2$ , a good source is braided monoidal categories, e.g. a category  $C$  with a functor  $\oplus C \times C \rightarrow C$ , and a natural transformation  $(- \oplus -) \oplus - \Rightarrow - \oplus (- \oplus -)$  from functors  $C \times C \times C \rightarrow C$ , called the *associator*, and a *commutator* which is a natural transformation whose components are  $x \oplus y \rightarrow y \oplus x$ , which we call the braiding. This satisfies some axioms, including pentagon and hexagon. If  $\sigma^2 = \text{id}$ , we call it a symmetric monoidal category.

For example,  $\text{Vect}_F$ , the category of vector spaces of dimension  $1 \leq \dim < \infty$  is a non-unital braided monoidal category, with  $\oplus$  from a direct sum. We will mostly care about non-unital  $E_2$ -algebras.

For such a category  $C$ , the classifying space  $BC$  has the structure of a (non-unital)  $E_2$ -algebra. We can write  $BC \simeq \coprod_x B\text{Aut}_C(x)$ , where we are taking one  $x$  in each isomorphism class.

In the vector space example, we can take  $B\text{Vect}_F = \coprod_{n \geq 1} B\text{GL}_n(F)$  is an  $E_\infty$  algebra, which gives algebraic  $K$ -theory.



**Example.** Surfaces (Miller, Fiedorowicz-Song) we have a category  $\Gamma$  whose objects are surfaces (compact connected oriented 2-manifold)  $\Sigma$  with  $\partial\Sigma = S^1$  a specified diffeomorphism. The morphisms are isotopy classes of diffeomorphisms rel boundary. We call the automorphisms of a surface of genus  $g$  as  $\Gamma_{g,1} = Mod_{g,1}$ . This has a braided monoidal structure obtained by gluing along the boundaries. This gives the direct sum operation. The braiding comes from taking a half-Dehn twist from gluing  $\Sigma$  then  $\Sigma'$  and extending it to a diffeomorphism to get the space which was obtained from gluing  $\Sigma'$  and then  $\Sigma$ . This is not symmetric monoidal, since the half Dehn twist squares to a full Dehn twist, which is not isotopic to the identity.

Then  $B\Gamma = \amalg B\Gamma_{g,1}$ , which becomes an  $E_2$ -algebra.<sup>6</sup>

“Built examples”: if  $Z$  is a space, then  $\mathcal{D}(Z) = \amalg_{n \geq 0} \mathcal{D}(n) \times Z^n/S_n$  is called the free  $E_2$ -algebra on  $Z$ .

*Cell attachment* let  $(X, \mu)$  be an  $E_2$ -algebra, and take  $e : \partial D^k \rightarrow X$  to be an attaching map, and we can take

$$\begin{array}{ccc} \mathcal{D}(\partial D^k) & \xrightarrow{e} & X \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{D}(D^k) & \longrightarrow & X', \end{array}$$

which is a pushout in  $E_2$ -algebras. Iterating this procedure, we get “cellular”  $E_2$ -algebras obtained from  $\emptyset$  by iterating cell attachment.

**Question:** Given an  $E_2$ -algebra  $C$  encountered in nature, how can we build a cellular  $E_2$ -algebra  $A$  and an  $E_2$  map  $A \xrightarrow{\simeq} C$  which shows that it is weakly equivalent to the one we encountered?

This is always possible, but we can ask the question: how many cells do we need?

If  $A$  is built by cell attachment, then it has a filtration given by the order in which we attached cells, so that the associated graded space  $\text{gr}(A)$  is isomorphic to  $\mathcal{D}(Z) = \vee \mathcal{D}(n)_+ \wedge Z^{\wedge n}/S_n$ , which is a free  $E_2$ -algebra in based spaces. Even more than this, it is a free  $E_2$ -algebra on a wedge of spheres. We get a spectral sequence

$$E^1 = H_*(\mathcal{D}(\vee_{\alpha} S_{\alpha}^n); k) \Rightarrow H_*(A; k) \cong H_*(X; k).$$

By F. Cohen,  $H_*(\mathcal{D}(Z); k)$  is known to be a functor of  $H_*(Z; k)$  whenever  $k$  is a field.

*Derived indecomposables:* If  $\mathcal{D}$  is an operad equivalent to the little disks operad and  $(X, \mu : \mathcal{D}(X) \rightarrow X)$  is an  $E_2$ -algebra, then the decomposables are  $\text{Dec}(X) = \text{im}(\amalg_{n \geq 2} \mathcal{D}(n) \times X^n/S_n \rightarrow X)$ . Then we define the indecomposables as

$$Q_{E_2}(X) = \text{Indec}(X) = X/\text{Dec}(X),$$

which is a pointed space.

Goal: (assuming  $\mathcal{D}(1) = \{*\}$ , show that  $\text{Indec}(\text{Free}(Z)) = Z \amalg \{*\}$  and pushouts are sent to pushouts). Then the built algebra is sent to a CW one with one ordinary cell

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<sup>6</sup> $B\Gamma_{g,1}$  is the surface bundle over  $M_{g,1}$ .

for each step. The problem is that  $X \xrightarrow{\sim} X'$  is not sent to a weak equivalence. So we take a derived functor  $Q_{E_2}^{\mathbb{L}}$ . Then in our setting above, we have

$$\text{Indec}(A) \xleftarrow{\sim} Q_{E_2}^{\mathbb{L}}(A) \xrightarrow{\sim} Q_{E_2}^{\mathbb{L}}(X).$$

This gives us a lower bound on the number of cells needed.

**Calculation:** (Getter-Jones 92, Basterra-Mandell, J. Francis, Lurie) gives a simplicial model for  $Q_{E_1}^{\mathbb{L}}$  and  $Q_{E_2}^{\mathbb{L}}$ . If  $X = BC$  then

$$\text{Dec}_{E_1}^{\mathbb{L}}(X) \simeq \text{hocolim}_{x \in C} |T^{E_1}x|,$$

where  $T^{E_1}(x)$  is a certain simplicial complex. If we take the example  $\text{Vect}_F$  or projective modules over a ring, we get something called the “split building” (Charney) for  $C = \text{Vect}_F$ .

## The Grothendieck ring of varieties, and stabilization in the algebro-geometric setting: Part 1

Ravi Vakil

References you should check out: Margaret Bilu’s thesis, Sean Howd and Margaret Bilu (upcoming article on arxiv).

The space we are considering tend to be moduli spaces, and come in families which stabilize (in some sense, usually homological) as a parameter tends to infinity. In smaller parameters of the moduli spaces, we often get accidents of small numbers, usually occurring in 3, 4, and 5. For example, the reason we can solve cubics are accidents about  $S^3$ . On the geometric side, if we are counting covers of  $\mathbb{CP}^1$ , e.g. double covers which are elliptic curves, the moduli space of degree  $d$  covers, when  $d$  is up to 5, is *unirational*, meaning that we can map onto it from some high dimensional affine space  $\mathbb{A}^D$  which hits all degree  $d$  covers. If we are looking at  $M_g$  when  $g$  is small, then it is also unirational, meaning we can describe curves of small genus; however for instance if  $g \geq 24$  then  $M_g$  is not unirational so we can’t describe a typical curve of this genus.

To motivate this we will do some computations. Let  $S = \{p, q, r\}$  be a finite set of three elements. How many ways can we have  $n$  objects among this set? We have a generating function, where the coefficient on  $t^j$  tells us how many ways to have  $j$  objects:

$$\begin{aligned} & 1 + (p + q + r)t + (p^2 + q^2 + r^2 + pq + pr + qr)t^2 + \dots \\ & = (1 + pt + p^2t^2 + \dots)(1 + qt + q^2t^2 + \dots)(1 + rt + r^2t^2 + \dots) \\ & = \frac{1}{(1 - pt)} \frac{1}{(1 - qt)} \frac{1}{(1 - rt)}. \end{aligned}$$

In general the generating function of complete symmetric functions is

$$\frac{1}{1 - e_1t + e_2t^2 + \dots + (-1)^m e_mt^m} = \sum_{p \in \mathbb{Z}} h_p t^p.$$

The binomial theorem tells us

$$(1 + t)^v = 1 + vt + \binom{v}{2}t^2 + \dots,$$

and we can prove things like

$$(1 + t)^v(1 + t)^w = (1 + t)^{v+w},$$

by counting things.

If  $V$  is a (say, finite-dimensional) vector space, we might think of  $\binom{V}{k}$  as  $\wedge^k V$ , and then we could write

$$\frac{1}{(1-t)^V} = 1 + Vt + \text{Sym}^2(V)t^2 + \text{Sym}^3(V)t^3 + \dots$$

By this logic, we might write  $\frac{1}{(1-pt)}$  from before as  $\frac{1}{(1-t)^p}$  and the generating function of elements from  $S$  as  $\frac{1}{(1-t)^{p+q+r}}$ .

If we let  $V = \mathbb{C}^n$  and take a representation of  $(\mathbb{C}^\times)^n$  we get an analog of the generating function from before. These polynomials should have coefficients living in  $K_0(G\text{-reps})$ .

Then we can think about what this means for varieties:

$$\frac{1}{(1-t)^{\mathbb{A}^1}} = \sum_{n \geq 0} \text{Sym}^n \mathbb{A}^1 t^n.$$

If we have  $n$  unordered numbers in  $\mathbb{A}^1$ , this is a polynomial, i.e. an element of  $\mathbb{A}^n$  so we can think of the generating function above as

$$\sum \mathbb{A}^n t^n = \sum (\mathbb{A}^1 t)^n = \frac{1}{1 - \mathbb{A}^1 t}.$$

For  $\mathbb{P}^1$ , we have

$$\sum \text{Sym}^n \mathbb{P}^1 t^n = \sum \mathbb{P}^n t^n,$$

and  $\mathbb{P}^1$  is  $\mathbb{A}^1$  with a point, so we can write

$$\frac{1}{(1-t)^{\mathbb{P}^1}} = \frac{1}{(1-t)^{\mathbb{A}^1}} \frac{1}{(1-t)^{pt}}.$$

We should think about things of this form as zeta functions  $\zeta_X(t) = \frac{1}{(1-t)^X}$ .

**Definition.** The *Grothendieck Ring of Varieties* over a field  $k$  is generated by  $[X]$ , where  $X$  is a variety over  $k$  modulo isomorphism, and it has an *additive relation* (cut and paste/scissor relation) for  $U \hookrightarrow X \hookleftarrow Z$  with  $Z = X - U$ , we have that

$$[X] = [U] + [Z].$$

And we have multiplication by

$$[X] \times [Y] = [X \times Y],$$

which has a unit given by a point. If this product is not reduced, we take its reduction so that it is a variety. Writing  $L = \mathbb{A}^1$ , we can write  $[\mathbb{P}^2] = L^2 + L + 1$ .

We can think about the real numbers as

$$[\mathbb{R}] = [\mathbb{R}^{>0}] + [\mathbb{R}^{<0}] + [pt],$$

hence  $[\mathbb{R}] = 2[\mathbb{R}] + 1$ , and  $[\mathbb{R}^k] = (-1)^k \dots$ , which looks like an Euler characteristic.

**Theorem** (Deligne). Given a smooth proper compact algebraic manifold over  $\mathbb{C}$ , we can compute all the Betti numbers, components, Hodge structures, just by its value in the Grothendieck ring. This is due to the theory of weights on cohomology.

Suppose  $k = \mathbb{F}_q$ , we get a map which counts points

$$K_0(\text{Var}_k) \xrightarrow{\#pts} \mathbb{Z},$$

which we note respects the addition and multiplication on the Grothendieck ring  $K_0(\text{Var}_k)$ .

We can think of a zeta function (defined by Kapranov)

$$\zeta_X(t) = \sum_{n \geq 0} [\text{Sym}^n X] t^n.$$

Applying  $\#pts$  to this, we get an element in  $\mathbb{Z}[[t]]$ , which is the Weil zeta function  $\zeta_X(t)$ . Thus the Weil zeta function is just counting points on symmetric powers.

Say we could equate  $X$  with an alternating sum of its cohomology, i.e. an Euler characteristic, then we would get

$$\frac{1}{(1-t)^X} = \frac{1}{(1-t)^{H_0 - H^1 + H^2 + \dots}} = \frac{(1-t)^{H^1 + H^3 + \dots}}{(1-t)^{H^0 + H^2 + \dots}}.$$

**Theorem** (Macdonald 1960). We have that

$$\sum \chi(\text{Sym}^n X) t^n = \frac{1}{(1-t)^{\chi(X)}}.$$

We will try to categorify this result. Taking the vector spaces  $H^i$ , putting them in an alternating order, we can view them as vector spaces with representation, and take the trace to get the Weil function. Then to prove the Weil conjectures, we can just show étale cohomology satisfies certain properties. So the Weil zeta function being rational comes from the fact that finitely many étale cohomological groups are nonzero.

## Hypersurfaces

We would like to count hypersurfaces appropriately. In Jordan's talk, we asked for a number of squarefree integers. Over  $\text{Spec } \mathbb{Z}$  this became  $\frac{1}{\zeta(2)}$ .

Bjorn Poonen's work shows that the probability that a given hypersurface is smooth, as  $d \rightarrow \infty$  is

$$\frac{1}{\zeta(\dim X + 1)},$$

where this is the Weil zeta function.

In the case of topology (Vassiliev, Tommasi) consider  $\mathbb{C}^n$  and all smooth functions on it. What are the odds of the zero set of a function being smooth?

In  $K_0(\text{Var}_k)$ , what is the probability that a hypersurface of degree  $d$  is smooth? This makes sense as  $d \rightarrow \infty$ , and we get that the limit becomes  $\frac{1}{\zeta(\dim X + 1)}$ , where this is the zeta function in the Grothendieck ring.

Morally: every time we say something in terms of zeta values, it should have interpretations in topology, arithmetic, and point counting over finite fields.

**Question:** Can we think about  $K_0$  of Galois representations?

### Conjectures, heuristics, and theorems in arithmetic statistics:

#### Part 1

Wei Ho

**Question:** How many number fields are there?

We can set discrete invariant to start counting, for example degree  $d$ , with Galois group  $G$ , etc. We then have to say how we want to order things, for number fields it is reasonable to order by discriminant, by *Hermite's Theorem* (we only have finitely many number fields of given degree up to a given discriminant). We can then talk about asymptotics as the discriminant changes.

**Conjecture** (Malle). Let  $G$  be a finite group with an embedding  $G \hookrightarrow S_n$ . Then

$$\lim_{x \rightarrow \infty} \frac{\#\{G\text{-number fields of discriminant } < x\}}{x^{1/a}(\log(x))^b}$$

exists and is nonzero, where  $n - a$  is the maximal number of orbits of  $g \in G \subseteq S_n$ .

This is known for abelian groups, nilpotent groups,  $S_3$ ,  $D_4$ ,  $S_5$ , etc.

**Heuristic for constant** (Bhargava) just multiply local factors. If we impose local conditions, we change the corresponding factor.

**Question:** What is the distribution of class groups of number fields?

**Cohen-Lenstra heuristic:** (1984) class groups behave as “random” finite abelian groups (weighting by  $\frac{1}{|\text{Aut}(G)|}$ ).

With this idea, we can make a lot of predictions: let  $h$  be the class number of an imaginary quadratic fields, and  $p \neq 2$ . Then

$$P(p|h) = 1 - \prod_{k \geq 1} (1 - p^{-k}).$$

For example:

$p = 3$	43.987%
$p = 5$	23.967%
$p = 7$	16.320%
$\text{Cl}_3 \cong \mathbb{Z}/9$	9.335%
$\text{Cl}_3 \cong (\mathbb{Z}/3)^2$	1.167%.

Cohen-Martinet (1990) worked on other types of number fields.

Friedman-Washington (1987) wanted to determine what happened in the function field case, by taking the following analogies:

$\mathbb{Q}$	$\mathbb{F}_q(t)$
quadratic fields $K$	hyperelliptic curves over $\mathbb{F}_q$
$\text{Cl}(K)_\ell$	the $\ell$ -Sylow subgroup $\text{Pic}^0(C)_\ell$
discriminant	genus $g$ .

But in the function field case, we now have Frobenius around, and it is a good idea to use it. They took an action of  $(\text{id} - \text{Frob})$  on the Tate module  $T_\ell(C) \cong \mathbb{Z}_\ell^{2g}$ . Thus we have some  $(2g \times 2g)$ -matrix lying around, and it turns out that its cokernel  $\text{coker}(\text{id} - \text{Frob}) = \text{Pic}^0(C)_\ell$  is exactly what we care about. Thus we can study the cokernel of such matrices and see if they are distributed randomly to check if this matches up with the Cohen-Lenstra heuristic.

Achter (2004) proved that this distribution actually is correct (for  $q \not\equiv 1 \pmod{\ell}$ ). We can think about this as good evidence that the analogy is useful.

Venkatesh-Elleberg (2010) in the number field case, we actually have a random matrix around as well— we have that  $\text{Cl}(K) = I(K)/P(K)$ , and we can take a finite set of primes  $S$  which generate the class group, and then we can write  $\text{Cl}(K) = I^S(K)/\mathcal{O}_S^*$  (where  $I^S(K)$  is a free abelian group on ideas in  $S$ , and  $\mathcal{O}_S^*$  is the  $S$ -units). Thus we can think of  $\text{Cl}(K)_\ell$  as the cokernel

$$\text{coker}(\mathcal{O}_S^* \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \rightarrow I^S(K) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell),$$

that is, an  $|S| \times |S|$  matrix over  $\mathbb{Z}_\ell$ . Then we can think about what happens if these matrices are behaving randomly.

We will now talk about analogies between number fields and elliptic curves.

Number fields $K$	elliptic curves $E/\mathbb{Q}$
roots of unity	$E(\mathbb{Q})_{tor}$
unit group $u(K)$	$E(\mathbb{Q})$
$ \text{disc} $	conductor
$\text{Cl}(K)$	$\text{III}(E)$
$R(K)$	$R(E)$
$0 \rightarrow U(K)/U(K)^p \rightarrow \text{Sel}_p(K) \rightarrow \text{Cl}(K)[p] \rightarrow 0$	$0 \rightarrow E(\mathbb{Q})/pE(\mathbb{Q}) \rightarrow \text{Sel}_p(E) \rightarrow \text{III}(E)[p] \rightarrow 0.$

**Remark.** We have maps

$$\begin{array}{ccc} E(\mathbb{Q})/nE(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q}, E[n]) \\ \downarrow & & \downarrow \beta \\ E(\mathbb{A})/nE(\mathbb{A}) & \xrightarrow{\alpha} & H^1(\mathbb{A}, E[n]), \end{array}$$

where  $E(\mathbb{A}) = \prod_v E(\mathbb{Q}_v)$  and  $H^1(\mathbb{A}, E[n]) = \prod'_v H^1(\mathbb{Q}_v, E[n])$ , and we define a *Selmer group* by

$$\text{Sel}_n(E) = \beta^{-1}(\text{im}(\alpha)) \subseteq H^1(\mathbb{Q}, E[n]).$$

And we can define

$$\text{III}(E) := \ker \left( H^1(\mathbb{Q}, E) \rightarrow \prod_v H^1(\mathbb{Q}_v, E) \right),$$

which is a torsion abelian group with alternating pairing.

Delaunay (2001) We can model III via ‘‘Cohen-Lenstra philosophy.’’ And we can model III[p] as a finite abelian  $p$ -group with alternating pairing.

Poonen-Rains (2012) modeled the Selmer group  $\text{Sel}_p$  as the intersection of random maximal isotropic subgroups in  $2n$ -dimensional quadratic spaces over  $\mathbb{F}_p$ .<sup>7</sup> Taking a limit as  $n \rightarrow \infty$ , they can make predictions about how Selmer groups behave.

BKLPR (2015) modeled the entire sequence

$$0 \rightarrow E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \rightarrow \text{Sel}_{p^\infty}(E) \rightarrow \text{III}[p^\infty] \rightarrow 0,$$

using the ideas of maximal isotropic subgroups.

In terms of rank, most people believe the *minimalist conjecture*: half of elliptic curves have rank 0, and half have rank 1.

PPVW (2016) have a random matrix model for ranks of elliptic curves. Ordering by height  $H$ , we have that

$$\#\{E : \text{rank} = i\} = \begin{cases} H^{20/24+o(1)} & i = 0, 1 \\ H^{(21-i)/24+o(1)} & 2 \leq i \leq 21. \end{cases}$$

We can look at data and see how it matches up with rank, and see how some of these ideas are proved.

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<sup>7</sup>As long as  $n$  is prime, this is the intersection of the images of  $\alpha$  and  $\beta$ , which are maximal isotropics.

## $\mathbb{A}^1$ -enumerative geometry: Part 2

Kirsten Wickelgren

Joint w Jesse Kass, Marc Levine, Jake Solomon. We will talk about  $\mathbb{A}^1$ -degree for counting rational curves.

Last time we talked about degrees of maps between spheres. Now we talk about degrees of maps between smooth schemes.

In algebraic topology, given  $f : X \rightarrow Y$  a map of smooth compact oriented dimension  $n$  manifolds. Then there is a fundamental class generating the homology  $H_n(X) \cong \mathbb{Z}[x]$ , and we can define the degree as the integer multiple of the pushforward in terms of the fundamental class of  $Y$ :

$$f_*[X] = \deg(f)[Y].$$

We compute this via local degrees as  $\deg f = \sum_{q \in f^{-1}(p)} \deg_q f$ . Back in  $\mathbb{A}^1$ -homotopy theory, Morel provided a degree map

$$\deg^{\mathbb{A}^1} : [\mathbb{P}^n / \mathbb{P}^{n-1}, \mathbb{P}^n / \mathbb{P}^{n-1}]_{\mathbb{A}^1} \rightarrow \text{GW}(k).$$

The degree has coherence with taking real and complex points via:

$$\begin{array}{ccccc} [S^n, S^n] & \xleftarrow{\mathbb{R}\text{-pts}} & [\mathbb{P}^n / \mathbb{P}^{n-1}, \mathbb{P}^n / \mathbb{P}^{n-1}]_{\mathbb{A}^1} & \xrightarrow{\mathbb{C}\text{-pts}} & [S^{2n}, S^{2n}] \\ \text{deg} \downarrow & & \downarrow \text{deg}^{\mathbb{A}^1} & & \downarrow \text{deg} \\ \mathbb{Z} & \xleftarrow{\text{sig}} & \text{GW}(k) & \xrightarrow{\text{rank}} & \mathbb{Z}. \end{array}$$

**Question:** Can we define a  $\text{GW}(k)$ -valued degree for  $f : X \rightarrow Y$  a map between smooth schemes? This was thought of by J. Fasel, F. Morel. The perspective is from KLS: to define  $\deg f = \sum_{q \in f^{-1}(p)} \deg_q f$ , we must

1. define a local degree
2. orientation ?
3. finite fibers?
4. independence of the choice of  $p$

For 1) we take a small ball around an isolated point in the fiber, and look at the induced map on local spheres via purity:  $Y/(Y - p) \simeq T_p Y / (T_p Y - 0) \simeq \mathbb{P}^n / \mathbb{P}^{n-1}$ .

We compare this with  $Jf \neq 0$ , given by  $Jf = \det \left( \frac{\partial f_j}{\partial x_j} \right) \in k(q)$ . Then we define

$$\deg_q f = \text{Tr}_{k(q)/k} \langle Jf(q) \rangle.$$

For 2), we take a map on tangent spaces  $Tf : TX \rightarrow TY$ , and we take the Jacobian to be

$$Jf = \det Tf \in \text{Hom}(\det TX, f^* \det TY)(X),$$



and since these schemes were smooth, these are line bundles. In order for the bracket  $\langle - \rangle$  to make sense, we only need to know the value of  $Jf$  up to squares. In other words, if we could make a trivialization of this line bundle which was well-defined up to squares, this could make sense.

**Definition.** We say  $f$  is *relatively oriented* if  $\text{Hom}(\det TX, f^* \det TY) \cong \mathcal{L}^{\otimes 2}$  is isomorphic to the square of a line bundle on  $X$ .

**Definition.** Bases of  $T_q X$  and  $T_p Y$  are *compatible* with the relative orientation if the associated element of  $\text{Hom}(\det TX, f^* \det TY)(q)$  is  $\ell \otimes \ell$  for some  $\ell \in \mathcal{L}(q)$ .

This makes 1) well-defined. Alternatively, a trivialization of  $\mathcal{L}$  identifies  $Jf_q$  with an element in  $\mathcal{O}_{X,q}$ , so we have a well-defined  $Jf_q \in \mathcal{O}_{X,q}/(\mathcal{O}_{X,q})^2$  and hence  $Jf(q) \in k(q)/(k(q)^\times)^2$ . Then we get  $\text{deg}_q f = \text{Tr}_{k(q)/k} \langle Jf(q) \rangle$ .

**Remark.** This will work in families.

We can show that 3) can be arranged on a complement of a codimension  $\geq 2$  subscheme of  $Y$  (i.e. we can chop out a codimension  $\leq 2$  subscheme and then get finite fibers) and we need  $Tf$  to be invertible at one point.

For 4) is this independent of  $p$ ? The answer is no— if we let  $C = \mathbb{C}/\mathbb{Z}[i]$  be a smooth elliptic curve over  $\mathbb{R}$ , then we can take a multiplication by 2 map to  $C$ . This has two components of real points. Over one component the real degree is 0 and over another component the real degree is 4, which alters the signature. However, there are great cases where the answer is yes.

**Theorem** (Harer). A family of symmetric, non-degenerate bilinear forms on finite-dimensional vector spaces over  $\mathbb{A}^1$  is (stably<sup>8</sup>) constant.

We say that  $Y$  is  $\mathbb{A}^1$ -*chain connected*<sup>9</sup> when the following equivalence relation identifies all points of  $Y$  with the same residue field: for any two points  $y_1, y_2 \in Y$  with  $k(y_1) = k(y_2)$ , we say  $y_1 \sim y_2$  when there exists  $u : \mathbb{A}^1 \rightarrow Y$  with  $t_1, t_2 \in \mathbb{A}^1$  so that  $u(t_i) = y_i$ .

**Theorem.** Let  $f : X \rightarrow Y$  be a map of smooth  $d$ -dimensional  $k$ -schemes such that  $Tf$  is invertible at a point and  $f$  is relatively oriented after possibly removing something of codimension  $\leq 2$  from  $Y$  and restricting to its complement, *and* we require that  $Y$  is  $\mathbb{A}^1$ -chain connected with a rational point. Then

$$\text{deg } f = \sum_{q \in f^{-1}(p)} \text{deg}_q f \in \text{GW}(k),$$

for any closed point with finite fiber. This is well-defined and independent of  $p$ .

This is computable, provided  $Jf \neq 0$ . We note that  $Jf \neq 0$  is definitely nonzero at the generic point  $q$ , so we could take

$$\text{deg } f = \text{Tr}_{k(X)/k(Y)} \langle Jf(q) \rangle \in \text{GW}(k(Y)),$$

and this actually lives in  $\text{GW}(k)$  provided the hypotheses above are met.

<sup>8</sup>We may have to add a direct summand. We can ignore this in characteristic  $\neq 2$ .

<sup>9</sup>Closely related to rationally connected.

**Example.** Take  $C = \{y^2 = p(z)\}$  an elliptic curve, and take the projection  $\pi : (z, y) \mapsto z$  down to  $\mathbb{P}^1$ . Taking  $\frac{dz}{2y} \mapsto 1$ , we get a trivialization  $TC^* \cong \mathcal{O}$ , and we can get  $(T\mathbb{P}^1)^* \cong \mathcal{O}(-1)^{\otimes 2}$  via  $dz$  maps to  $|z|$  then we can see

$$\pi^*(dz) = dy \left( \frac{dz}{2y} \right),$$

and we check that  $J\pi$  at the generic point is  $2y$ . Finally, taking the trace, we get

$$\mathrm{Tr}_{k(C)/k(z)} \langle 2y \rangle = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} = \mathbb{H},$$

in the basis  $1, 1/y$ . Thus  $\mathrm{deg}(\pi) = \mathbb{H} \in \mathrm{GW}(k)$ .

**Motivating example:** Counting rational curves in  $\mathbb{P}^2$ .

A rational curve of degree  $d$  is  $u : \mathbb{P}^1 \rightarrow \mathbb{P}^2$  of the form  $[u_0(t), u_1(t), u_2(t)]$  where each  $u_i$  is homogeneous of degree  $d$ . Taking  $p_1, \dots, p_n \in \mathbb{P}^2$ , we can ask how many rational curves pass through all these points.<sup>10</sup> There are finitely many when  $n = 3d - 1$ :

$d$	$3d - 1$	number of rational curves
1	2	1
2	5	1
3	8	12
4	11	620.

Over a field  $k$ , we compare with M. Levine’s “Welschinger invariants.” There is a Kontsevich moduli space  $\overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2, d)$  parametrizing stable maps  $u$  from a genus 0 curve to  $\mathbb{P}^2$  together with  $p_1, \dots, p_{3d-1}$  smooth points. we have an evaluation map

$$\mathrm{ev} : \overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2, d) \rightarrow (\mathbb{P}^2)^{3d-1},$$

which evaluates the maps. For any  $\sigma : \mathrm{Gal}(\overline{k}/k) \rightarrow \Sigma_{3d-1}$ , we define descent data on  $(\mathbb{P}^2)_{\overline{k}}^{3d-1}$  by  $g \in \mathrm{Gal}(\overline{k}/k)$  acts by  $g\sigma(g)$ . Thus we obtain a twist  $(\mathbb{P}^2)_{\sigma}^{3d-1}$  which is really a restriction of scalars of  $\mathbb{P}^2$  over residue fields  $\overline{k}^{\mathrm{Stab}(i)}$ . Similarly, we twist the evaluation map by  $\sigma$  and obtain

$$\mathrm{ev}_{\sigma} : \overline{\mathcal{M}}_{0,3d-1}(\mathbb{P}^2, d)_{\sigma} \rightarrow (\mathbb{P}^2)_{\sigma}^{3d-1}.$$

So we obtain curve-counting invariants  $N_{d,\sigma} := \mathrm{deg} \mathrm{ev}_{\sigma}$ .

**Question:** Can we get a degree from cohomology?

**Answer:** When you are relatively oriented, we can get a pushforward in oriented Chow. It is maybe untrue that oriented Chow of the base is generated by a fundamental class.

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<sup>10</sup>E.g. if we had two points and  $d = 1$  there is a unique line through any two points.

## Coincidences of homological densities, predicted by arithmetic: Part 2

Benson Farb

Joint with Jesse Wolfson, Melanie Wood.

**A topological coincidence:** Consider the two spaces with associated theorems about homological stability

1.  $\text{Poly}_n = \{F \in \mathbb{C}[x] \text{ monic squarefree of degree } n\}$ .

**Theorem** (Arnol'd 1969). For all  $i \geq 0$ , we have that

$$\lim_{n \rightarrow \infty} H_i(\text{Poly}_n; \mathbb{Z}) = H_i(\Omega^2 \mathbb{C}P^1; \mathbb{Z}).$$

2. Let the space of degree  $n$  holomorphic maps be  $\text{Hol}_n^* = \{f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}} : \deg f = n, f(\infty) = 1\}$ .

This is the same as:

$$\begin{aligned} &= \left\{ \frac{f}{g} : f, g \in \mathbb{C}[x], \deg f = \deg g = n, \text{ monic, no common roots} \right\} \\ &= \{\text{disjoint pairs of } n\text{-tuples of points in } \mathbb{C}\}. \end{aligned}$$

This last equality is by looking at roots and poles. Given a set of roots and poles, it determines the rational map purely by this topological information.

**Theorem** (Segal, 1979). For all  $i \geq 0$ , we have that

$$\lim_{n \rightarrow \infty} H_i(\text{Hol}_n^*; \mathbb{Z}) = H_i(\Omega^2 \mathbb{C}P^1).$$

**Note.** We have that  $\pi_1(\text{Poly}_n) = \text{Br}_n$  is the braid group on  $n$  strands, and  $\pi_1(\text{Hol}_n^*) = \mathbb{Z}$ . So we see they are quite different.

Melanie Wood: “Number theorists could have predicted this.” There is a well-known analogy going back to André Weil associating  $\text{Poly}_n$  and the set  $\{r \in [1, n] : r \text{ squarefree}\}$ . Additionally under this analogy,  $\text{Hol}_n^*$  corresponds to  $\{(r, s) \in [1, n]^2 : \gcd(r, s) = 1\}$ . Then if we take the limit of probabilities:

$$\begin{aligned} \lim_{n \rightarrow \infty} \text{Prob}(r \in [1, n] \text{ squarefree}) &\approx \prod_p \left(1 - \frac{1}{p^2}\right) = \zeta(2)^{-1} = \frac{6}{\pi^2} \\ \lim_{n \rightarrow \infty} \text{Prob}((r, s) \in [1, n]^2 : \gcd(r, s) = 1) &\approx \prod_p \left(1 - \frac{1}{p} \cdot \frac{1}{p}\right) = \zeta(2)^{-1}. \end{aligned}$$

We note that there are similar counts for number fields, not just  $\mathbb{Q}$ . Thus in the analogies above,  $\Omega^2 \mathbb{C}P^{n-1}$  corresponds to  $\zeta(n)^{-1}$ .

**Classical example:** For all  $m, n \geq 1$ , we have that

$$\lim_{d \rightarrow \infty} \frac{\#\{(a_1, \dots, a_m) \in [1, d]^m : \gcd(a_1, \dots, a_m) \text{ is } n\text{-power free}\}}{\#[1, d]^m} = \zeta(mn)^{-1}.$$

Does there exist a topological incarnation of this? These correspond to *spaces of 0-cycles*. The setup:  $X$  is a connected, smooth, oriented manifold, and  $\dim H^*(X; \mathbb{Q}) < \infty$ . We have  $m, n \geq 1$  and  $\vec{d} = (d_1, \dots, d_m)$  with each  $d_i \geq 0$ . Then we let

$$\mathrm{Sym}^{\vec{d}}(X) = \prod_{i=1}^m \mathrm{Sym}^{d_i} X.$$

This has  $d_i$  labeled points (not necessarily distinct) of each color  $i$ .

Let  $Z_n^{\vec{d}}(X) \subseteq \mathrm{Sym}^{\vec{d}} X$  be the space where we allow points to come together, but we should not allow  $\geq n$  points of every color to come together at a single point.

- $Z_2^d(X) = U\mathrm{Conf}_d(X)$
- $Z_1^{d_1, \dots, d_m}(X) = \mathrm{Hol}_d^*(\mathbb{C}P^1, \mathbb{C}P^{m-1})$  sending  $[x, y] \mapsto [f_1(x, y) : \dots : f_m(x, y)]$

**Recall:** The *Poincaré polynomial* of a space  $X$  is the generating function of the rational Betti numbers:

$$P_X(t) = \sum_{i \geq 0} \dim H_i(X; \mathbb{Q}) t^i \in \mathbb{Z}[[t]].$$

We can try the next simplest space  $X = \mathbb{C} - \{0\}$ , and we get that

$$\begin{aligned} \lim_{d \rightarrow \infty} P_{Z_2^d(X)}(t) &= 1 + 2t + 2t^2 + 2t^3 + \dots \\ \lim_{d \rightarrow \infty} P_{Z_1^{d,d}(X)}(t) &= 1 + 3t + 4t^2 + 4t^3 + \dots \end{aligned}$$

“Take Weil more seriously:” forgot to divide by the ambient space. Explicitly, we recall that we had to divide by  $\#[1, d]^m$  in the  $n$ -power example, so we need an analogy of this in the topological context.

Let  $P_{\mathrm{Sym}^d(\mathbb{C}^*)}(t) = 1 + t$  for all  $d \geq 2$  and we get

$$\begin{aligned} \lim_{d \rightarrow \infty} \frac{P_{Z_2^d(\mathbb{C}^*)}(t)}{P_{\mathrm{Sym}^d(\mathbb{C}^*)}(t)} &= \frac{1 + 2t + 2t^2 + \dots}{1 + t} = 1 + t + t^2 + t^3 + \dots \\ \lim_{d \rightarrow \infty} \frac{P_{Z_1^{d,d}}(t)}{P_{\mathrm{Sym}^d(\mathbb{C}^*) \times \mathrm{Sym}^d(\mathbb{C}^*)}(t)} &= \frac{1 + 3t + 4t^2 + \dots}{(1 + t)^2} = 1 + t + t^2 + t^3 + \dots \end{aligned}$$

**Theorem.** Let  $X$  be connected, oriented, smooth manifold with  $\dim H^*(X; \mathbb{Q}) < \infty$ . Assume

- (\*) the cup product of any  $k$  classes in  $H_c^*(X; \mathbb{Q})$  equals zero.

Then for any  $m, n \geq 1$  with  $mn \geq k$ , and  $\vec{d} = (d_1, \dots, d_m)$  with each  $d_i \geq 0$ , then

$$\lim_{\vec{d} \rightarrow \infty} \frac{P_{Z_n^{\vec{d}}(X)}(t)}{P_{\mathrm{Sym}^{\vec{d}}(X)}(t)} \in \mathbb{Z}[[t]],$$

exists and depends only on  $mn$  (and of course on  $\dim X$  and the Betti numbers  $b_i(X)$ ).

**Examples of  $X$  satisfying (\*):**

1.  $X$  a smooth affine variety over  $\mathbb{C}$  with  $mn > 2$  (this is sharp)
2. If  $X \subseteq \mathbb{C}^r$  a Zariski open subset of  $\mathbb{C}^r$  with  $r \geq 2$
3.  $X$  noncompact manifold,  $mn > \dim X$ .

**Remarks:**

0. The limit statement comes from (rational) homological stability for lots of examples
1. For fixed  $m \cdot n$  we get many coincidences
2. Meaning of division of Poincaré polynomials: there is an interpretation using factorization homology (Quoc Ho - factorization homology)
3. We get similar results for Euler characteristic (Hodge-Deligne polynomials).

**Open problems:**

1. When the statement fails, e.g. for the punctured torus, it is because some differential in the Leray SS doesn't vanish (the hypotheses above are used to make differentials vanish) thus there should be a correspondence between differentials and correction terms. What are these correction terms on the number field side?
2. Find other stories starting with other coincidences in arithmetic.

*Proof cartoon:*

- We go to an ordered case  $\widetilde{Z}_n^{\vec{d}}(X)$  which has an action of  $S_{d_1} \times \cdots \times S_{d_m}$
- Do the “local” case  $X = \mathbb{R}^n$  (Nir Gadish)
- Leray SS  $\widetilde{Z}_n^{\vec{d}}(X) \subseteq X^{|\vec{d}|}$
- Key ingredient: Björner-Wachs theory of lexicographical shellability.

 **$E_2$ -algebras in spaces: Part 2**

Søren Galatius

Setup:  $\mathcal{C}$  is a monoidal (resp. braided monoidal, resp. symmetric monoidal) category. Then  $X = B\mathcal{C}$  is a non-unital  $E_1$  (resp.  $E_2$ , resp.  $E_\infty$ ) space.

Reference: Fiedorowicz: “symmetric bar construction.”

We have that  $\sigma : x \oplus y \Rightarrow y \oplus x$  is a natural transformation between functors  $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ , and Fiedorowicz associates a map  $\mathbb{Z} \times \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  which gives

$$B\mathbb{Z} \times B\mathcal{C} \times B\mathcal{C} = S^1 \times B\mathcal{C} \times B\mathcal{C} \rightarrow B\mathcal{C}.$$

We gave a simplicial formula for  $Q_{\mathbb{L}}^{E_1} X, Q_{\mathbb{L}}^{E_2} X$ . Take  $\text{hocolim} : \text{Fun}(\mathcal{C}, s\text{Set}) \rightarrow s\text{Set}$  sending  $\mathbf{t} \mapsto B\mathcal{C}$ , where  $\mathbf{t}$  is the terminal functor sending everything to a point.

We want to calculate  $T^{E_1}, T^{E_2} \in \text{Fun}(\mathcal{C}, s\text{Set})$ . We get that  $T^{E_1}(x)$  has vertices of the form  $(x_0, x_1, x_1 \oplus x_1 \xrightarrow{\sim} x)$  modulo isomorphisms of triples. A one-simplex is  $(x_0, x_1, x_2, x_0 \oplus x_1 \oplus x_2)$  modulo isomorphisms, etc.

**Theorem.** If  $\text{Aut}_{\mathcal{C}}(x) \times \text{Aut}_{\mathcal{C}}(y) \xrightarrow{\oplus} \text{Aut}_{\mathcal{C}}(x \oplus y)$  is an injection of groups for each  $x, y \in \mathcal{C}$ , then

$$\text{hocolim}_{x \in \mathcal{C}} |T^{E_1}(x)| \simeq \text{Dec}_{\mathbb{L}}^{E_1}(X).$$

If we want indecomposables instead, we take the pointed spaces

$$\text{hocolim}_{x \in \mathcal{C}} \Sigma |T^{E_1}(x)| \simeq Q_{\mathbb{L}}^{E_1} X.$$

**Example.** Say  $\Gamma$  is the category of surfaces, whose morphisms are given relative the boundary  $\partial\Sigma$ . Then we have that  $T^{E_1}(\Sigma)$  has vertices which are triples  $\Sigma_0, \Sigma_1$  and a map from a space with  $\Sigma_0$  and  $\Sigma_1$  glued in down to  $\Sigma$  modulo isomorphism, which in this category is diffeomorphism relative to isotopy on the boundary. These correspond to separating arcs (up to isotopy) between two specified points on  $\partial$  dividing the surface so that it has positive surface on each side.

**Theorem.** We have that

$$|T^{E_1} \Sigma_{g,1}| \simeq \begin{cases} \vee S^{g-2} & g \geq 2 \\ \emptyset & g = 1 \end{cases}.$$

This high connectivity tells us how to build  $B\Gamma \simeq \coprod_{g \geq 1} B\Gamma_{g,1}$  as an  $E_1$ -space up to weak equivalence. We only need cells whose dimension is  $g - 1$  or above.

Let  $\mathcal{C} = \text{Vect}_F$  be a category of finite-dimensional vector spaces, not including vector spaces of dimension 0. Then for  $V \in \mathcal{C}$ , we have that  $T^{E_1}(V)$  has vertices  $(V_0, V_1, V_0 \oplus V_1 \xrightarrow{\cong} V)$  modulo isomorphism. This is in bijection with subspaces with their orthogonal complement. We think of this as  $V_0 \hookrightarrow V$  with a retraction whose kernel is  $V_1$ . The  $p$ -simplices are of the form

$$0 \subseteq V_0 \subseteq V_1 \subseteq \dots \subseteq V,$$

with retracts in each direction. This is the flag with a choice of splitting. Charney’s “split building.” proved that this is highly connected, that is,

$$|T^{E_1}(F^n)| \simeq \begin{cases} \vee S^{n-2} & n > 1 \\ \emptyset & n = 1. \end{cases}$$

This is related to  $T^{E_1}(V) \rightarrow T(V)$ , where  $T(V)$  is the Tits building. This is obtained by forgetting the splitting. We have that  $T(V)$  is also highly connected, and we get a map

$$|T^{E_1}(V)| \rightarrow |T(V)| = \vee S^{n-2},$$

where  $n = \dim V$ . The homology is  $\widetilde{H}_{n-2}(T(F^n)) = \text{St}_n$ , called the *Steinberg module*. This comes equipped with an action of  $\text{GL}_n(F)$ . Taking homology of  $T^{E_1}$  we get a *split Steinberg module* which comes with a map

$$\widehat{\text{St}}_n = \widetilde{H}_{n-2}(|T^{E_1}(F^n)|) \rightarrow \widetilde{H}_{n-2}(T(F^n)) = \text{St}_n.$$

We now get

$$H_*(Q_{\mathbb{L}}^{E_1}(X)) = \bigoplus_{n,d} H_{d-(n-1)}(\text{BGL}_n(F); \widehat{\text{St}}_n).$$

**Proposition.** If  $F$  is infinite, the kernel of the map  $\widehat{\text{St}}_n \rightarrow \text{St}_n$  is acyclic, i.e. the canonical map is an isomorphism:

$$H_*(\text{BGL}_n(F); \widehat{\text{St}}_n) \xrightarrow{\cong} H_*(\text{BGL}_n(F); \text{St}_n).$$

This latter term measures  $E_1$ -cells. Thus in rank  $n$  you only need  $n-1$  cells and above. Then in degree  $n$ , the  $n-1$  cells you need are exactly  $H_0(\text{GL}_n(F); \text{St}_n)$ .

How does this change for  $E_2$ ?

**Simplicial formula for  $Q_{\mathbb{L}}^{E_2} X$ :** Let  $X = B\mathcal{C}$ . Then we get

$$Q_{\mathbb{L}}^{E_2} X = \text{hocolim}_{x \in \mathcal{C}} \Sigma |T^{E_2}(X)|.$$

If  $\mathcal{C} = \text{Vect}_F$  as before, then we get  $T^{E_2}$  as a subcomplex of the join:

$$T^{E_2}(V) \hookrightarrow T^{E_1}(V) * T^{E_1}(V).$$

Then the pair  $((v_0, \dots, v_{p+1}), (v'_0, \dots, v'_{q+1})) \in T^{E_1} * T^{E_1}$  (with  $v_i \in V_i$  and  $v'_i \in V'_i$ ) live in  $T^{E_2}$  if they arise as splittings  $\bigoplus V_{i,j} = V$  with  $1 \leq i \leq q+1$  and  $1 \leq j \leq p+1$  and we have that

$$\begin{aligned} \bigoplus_i V_{i,j} &= V'_j \\ \bigoplus_j V_{i,j} &= V_i. \end{aligned}$$

**Theorem.** The pointed homotopy colimit  $\text{hocolim}_{V \in \text{Vect}_F} \Sigma T^{E_2}(V)$  is a model for the  $E_2$ -indecomposables  $Q_{\mathbb{L}}^{E_2}(X)$ .

**Theorem.** If  $F$  is infinite, then the inclusion into the join

$$|T^{E_2}(V)| \hookrightarrow |T^{E_1}(V)| * |T^{E_1}(V)|,$$

becomes an isomorphism after taking homotopy colimits then homology.

Note if we intend to take homology, we could look at  $|T(V)| * |T(V)|$  which is a wedge sum of spheres, and its homology is the Steinberg module tensored itself. Then

$$H_*(Q_{\mathbb{L}}^{E_2} X) = \bigoplus_{d,n} H_{d-(2n-2)}(\text{BGL}_n(F); \text{St}_n \otimes \text{St}_n).$$

These summands on the right measure  $E_2$ -cells. Again as above, we see that for rank  $n$ , we only need  $2n - 2$  cells and higher for  $E_2$  cells of  $\Pi_n BGL_n(F)$ . Moreover, the  $(2n - 2)$ -cells we need are exactly  $H_0(BGL_n(F); St_n \otimes St_n)$ .

We know what this group is.

**Theorem.** We have that

1.  $H_0(BGL_n(F); St_n \otimes St_n) \cong \mathbb{Z}$ .
2. there is a pairing  $St_n \otimes St_n \rightarrow H_0(GL_n(F); St_n \otimes St_n) \cong \mathbb{Z}$  which is symmetric, positive-definite, and non-degenerate after applying  $-\otimes k$  over  $k$  any field.

**Corollary.** We have that  $St_n \otimes \mathbb{Q}$  is indecomposable as a representation.

## Conjectures, heuristics, and theorems in arithmetic statistics:

### Part 2

Wei Ho

Last time:

- interested in distributions/statistics/asymptotics for arithmetic/algebraic objects
- try to model them — as linear algebra objects
- prove theorems about distribution/statistics/asymptotics for *random* such linear algebra objects
- make predictions/conjectures about the original objects.

Today we will talk about how to actually prove theorems.

**Example theorems:**

**Theorem** (Gauss conjecture, Lipschitz/Martens, 1860s). Let  $Cl(d)$  denote the class group of the quadratic ring with discriminant  $d$ . Then:

$$\sum_{-x < d < 0} \#Cl(d) \sim c \cdot x^{3/2},$$

where the constant is about  $\frac{\pi}{18\zeta(3)}$  for rings.

**Theorem** (Davenport-Heilbronn 1971). We have that

$$\begin{aligned} \#\{\text{cubic fields with discriminant } 0 < d < X\} &\sim \frac{1}{12\zeta(3)} X \\ \#\{\text{cubic fields with discriminant } 0 > d > -X\} &\sim \frac{1}{4\zeta(3)} X. \end{aligned}$$

The average size of 3-torsion subgroups of  $Cl$  for quadratic fields is  $\frac{4}{3}$  if  $d > 0$  and 2 if  $d < 0$ .



**Theorem** (Bhargava-Shakar, 2010s). The average size of the 2-Selmer group of elliptic curves over  $\mathbb{Q}$  (works over global fields) ordered by height<sup>11</sup> we have that

$$\text{avg}_{E/\mathbb{Q}} |\text{Sel}_2(E)| = 3.$$

For  $\text{Sel}_3$ , this is 4, for  $\text{Sel}_4$  this is 7, and for  $\text{Sel}_5$  this is 6.

This theorem provides an immediate corollary:

**Theorem.** The average rank is bounded. In particular, the average rank is bounded by 0.885.

*Proof idea for Davenport-Heilbronn:* Count points on a moduli space for whatever objects (will actually be a stack, not a scheme).

1. Find an explicit description of moduli space.

We can think about cubic rings as corresponding to a degree 3 subscheme of  $\mathbb{P}_{\mathbb{Z}}^1$ . This corresponds to a binary cubic form  $ax^3 + bx^2y + cxy^2 + dy^3$ , where  $a, b, c, d \in \mathbb{Z}$ . These live in  $\text{Sym}^3(\mathbb{Z}^2)/\text{GL}_2(\mathbb{Z})$  (we will write this  $V/G$ ), which is the correct moduli space to look at.

2. Count points on this orbit space.

Specifically, we want to count integral points  $V(\mathbb{Z})/G(\mathbb{Z})$ . We also need to worry about how we want to order things. Here we use the discriminant<sup>12</sup>.

3. Take  $G(\mathbb{Z})$  acting on the real points  $V(\mathbb{R})$ . We see that  $V(\mathbb{R})$  looks approximately like  $G(\mathbb{R})$ , since it has this one polynomial invariant<sup>13</sup>. Now we can think about  $G(\mathbb{Z})$  acting on  $G(\mathbb{R})$ , for which we have a fundamental domain  $\mathcal{F}$ . Then we want to “count” (up to discriminant) lattice points in  $\mathcal{F}$  using geometry of numbers, i.e. we estimate its volume.
4. We have to deal with cusps and “bad” points.
5. Finally, we may have to sieve to deal with local conditions.

Using class field theory, we can relate cubic rings with certain local conditions with order 3 elements in the class group of quadratic fields. Since we know how to count quadratic fields, we just look at the fiber of:

$$\{\text{order 3 elts. in Cl of quad. fields}\} \rightarrow \{\text{quad. fields}\}.$$

In the Selmer case, we have  $\{p\text{-Selmer elts. of } E\} \subseteq \{\text{genus 1 curve } C \text{ and degree } p \text{ line bundles } E \cong \text{Jac}(C)\}$  so we look at the fiber of:

$$\{\text{genus 1 curve } C \text{ and degree } p \text{ line bundles } E \cong \text{Jac}(C)\} \rightarrow \{E\}.$$

<sup>11</sup>For  $y^2 = x^3 + Ax + B$ , with  $A, B \in \mathbb{Z}$ , the height is  $\max\{4|A|^3, 27|B|^2\}$ .

<sup>12</sup>We have that  $\text{SL}_2(\mathbb{Z})$  acts on this space of polynomials, and polynomial invariants are generated by one invariant, the discriminant. Thus we have one polynomial invariant we are forced to count by. We could count by non-polynomial invariants e.g. the Julia invariant, but this is much harder.

<sup>13</sup> $G(\mathbb{C})$  acting on  $V(\mathbb{C})$  has one orbit, thus they look very similar.

## The Grothendieck ring of varieties, and stabilization in the algebro-geometric setting: Part 2

Aaron Landesman

### How many covers of $\mathbb{P}^1$ are there?

**Definition.** The Hurwitz space  $H_{d,g}$  parametrizes smooth proper geometrically connected curves  $C \xrightarrow{\pi} \mathbb{P}^1$ , genus of  $C$  is  $g$  and  $\deg \pi = d$ , and with Galois group of the closure  $S_d$ .

The simply branched Hurwitz space  $H_{d,g}^0 \subseteq H_{d,g}$  parametrizes those same curves where  $\pi$  is also simply branched — that is, every fiber is smooth or has one ramification point of order 2.

Context	Function fields $\mathbb{F}_q(t)$	Number fields	Cohomology	Grothendieck ring
Question	How many covers of $\mathbb{P}_{\mathbb{F}_q}^1$ ? That is, what is the size of $Hur(\mathbb{F}_q)$ ?	How many covers of $\text{Spec } \mathbb{Z}$ ?	What is $H_*(H_{d,g})$ ?	What is $[H_{d,g}]$ ?
What do we know?	In $d = 3$ , Datskovsky Wright, $d \leq 5$ Bhargava-Shankar-Wang, $d = 3$ Zhao, Gunther, $d = 3, 4$ Gunther, Hast, Matei	$d = 3$ Davenport-Heilbronn, $d \leq 5$ Bhargava	$d = 3$ stabilizes, Ellenberg-Venkatesh-Westerland	$d \leq 5$ Landesman-Vakil-Wood.

**Question:** Let  $d \geq 2$  and assume  $\text{char}(\mathbb{F}_q) \nmid d!$ . What is the limit<sup>14</sup>

$$\lim_{g \rightarrow \infty} \frac{[H_{d,g}^0]}{\mathbb{L}^{\dim H_{d,g}^0}} \stackrel{?}{=} \lim_{g \rightarrow \infty} \frac{[\text{Conf}_{2g+2d-2}(\mathbb{P}^1)]}{\mathbb{L}^{\dim H_{d,g}}} = 1 - \mathbb{L}^{-2}.$$

**Question:** Do we have that

$$\lim_{g \rightarrow \infty} \frac{[H_{d,g}]}{\mathbb{L}^{\dim H_{d,g}}} \stackrel{?}{=} \frac{1}{1 - \mathbb{L}^{-1}} \left( (1 - \mathbb{L}^{-1}) \left( \sum_{\lambda \vdash d} \mathbb{L}^{d-|\lambda|} \right) \right)^{\mathbb{P}^1}.$$

In order to make sense of this we work not in  $K_0(\text{Var}_k)$  but in  $K_0(\text{Var}_k) \left[ \mathbb{L}^{-1}, \{(\mathbb{L}^n - 1)^{-1}\}_{n \geq 1} \right]$ . And here “limit” means that  $X_n \rightarrow Y$  if  $\dim(X_n - Y) \rightarrow \infty$ .

**Theorem** (Landesman-Vakil-Wood). These questions are true for  $d \leq 5$ .

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<sup>14</sup>In the Grothendieck ring,  $\mathbb{L} = [\mathbb{A}^1]$ .

**Notation:** We will denote  $(1 - \mathbb{L}^{-2})^{\mathbb{P}^1} = \prod_{x \in \mathbb{P}^1} (1 - \mathbb{L}^{-2})$ .<sup>15</sup>

**Understanding how to count smooth curves:** Recall that Weil zeta functions are defined as

$$\frac{1}{Z_X(q^{-s})} = \prod_{\substack{x \in X \\ x \text{ closed}}} (1 - q_x^{-s}),$$

and we can write

$$\frac{1}{Z_X(\mathbb{L}^{-s})} = \prod_{x \in X} (1 - \mathbb{L}^{-s}).$$

**Lemma.** We have that<sup>16</sup>

$$\lim_{d \rightarrow \infty} \frac{\#\{\text{smooth elts of } H^0(\mathbb{P}^1, \mathcal{O}(d))\}}{\#H^0(\mathbb{P}^1, \mathcal{O}(d))} = \frac{1}{Z_{\mathbb{P}^1}(q^{-2})}.$$

**Idea 1:**

- (A) What is the chance that  $f \in H^0(\mathbb{P}^1, \mathcal{O}(d))$  is smooth at  $x$ ? We write  $f$  locally as  $f = f(x) + \varepsilon f'(x)$ . This is singular if  $f(x) = f'(x) = 0$ , which happens with probability  $\frac{1}{q_x^2}$ . Thus the chance of smoothness is  $1 - q_x^{-2}$ .
- (B) These chances are independence, so the chance of smoothness, which is the ratio written in the above lemma, is

$$\prod_{x \in \mathbb{P}^1} (1 - q_x^{-2}) = \frac{1}{Z_{\mathbb{P}^1}(q^{-2})}.$$

**Idea 2:** Inclusion-exclusion.

**Theorem.** We have that

$$\lim_{d \rightarrow \infty} \frac{[\text{smooth } f \in H^0(\mathbb{P}^1, \mathcal{O}(d))]}{\mathbb{L}^{d+1}} = \frac{1}{Z_{\mathbb{P}^1}(\mathbb{L}^{-2})}.$$

**Notation by example:** Let  $S_{2,2,1}$  denote polynomials with two blue marked points, two green marked points, one red marked point, and which is singular exactly at these points. We let  $S_{\geq 2,2,1}$  denote polynomials with the same marked points at above, and the polynomial is singular at these points and possibly elsewhere.

In order to set up inclusion-exclusion, we note that

$$\begin{aligned} \{\text{smooth sections}\} &= \{\text{all sections}\} \setminus \cup_{n \geq 1} \{\text{sections singular at exactly } n \text{ points}\} \\ &= \text{all sections} - S_1 - S_2 - \dots \end{aligned}$$

<sup>15</sup>This is analogous to  $\prod_{x \in \mathbb{P}^1} (1 - q_x^{-2})$  where  $q_x = \#k(x)$ .

<sup>16</sup>Think of  $H^0(\mathbb{P}^1, \mathcal{O}(d))$  as degree  $d$  squarefree homogeneous polynomials in 2 variables.

We note that we can write  $S_1 = S_{\geq,1} - S_{1,1} - S_{1,2} - \dots$  thus

$$\{\text{smooth}\} = \{\text{all}\} - S_{\geq,1} + S_{\geq,1,1} - S_{\geq,2}, \dots$$

Putting this together, we get<sup>17</sup>

$$\begin{aligned} [\text{smooth polys}] &= \sum_{\lambda} (-1)^{\#\text{colors in } \lambda} S_{\geq, \lambda} \\ &= \sum_{\lambda} (-1)^{\#\text{colors}} \text{Conf}_{\lambda}(\mathbb{P}^1) \mathbb{L}^{d+1-2 \cdot \#\{\text{sing. pts}\}} \\ &= \sum_{\lambda} (-1)^{\#\text{colors}} \text{Sym}_{\lambda}(\mathbb{P}^1) \mathcal{L}^{d+1-2 \cdot \#\{\text{sing. pts}\}} \\ &= \frac{\mathcal{L}^{d+1}}{Z_{\mathbb{P}^1}(\mathcal{L}^{-2})}. \end{aligned}$$

**Counting hyperelliptic curves:** We have that<sup>18</sup>

$$\begin{aligned} \lim_{g \rightarrow \infty} [H_{2,g}] &:= \lim_{g \rightarrow \infty} [\text{Conf}_{2g+2}(\mathbb{P}^1)] \\ &= \frac{[\text{smooth sections } H^0(\mathbb{P}^1, \mathcal{O}(2g+2))]}{\mathbb{L} - 1} \\ &= \frac{1}{1 - \mathbb{L}} \mathbb{L}^{2g-3} [\text{proportion of smooth}] \\ &= \frac{\mathbb{L}^{2g-3}}{\mathbb{L} - 1} \prod_{x \in \mathbb{P}^1} [\text{chance of smooth at } x] \\ &= \frac{\mathbb{L}^{2g-3}}{\mathbb{L} - 1} \prod_{x \in \mathbb{P}^1} (1 - \mathbb{L}^{-2}) \\ &= \frac{\mathbb{L}^{2g-3}}{\mathbb{L} - 1} \cdot \frac{1}{Z_{\mathbb{P}^1}(\mathbb{L}^{-2})} \\ &= \frac{\mathbb{L}^{2g-3}}{\mathbb{L} - 1} (1 - \mathbb{L}^{-2}) (1 - \mathbb{L}^{-1}) \\ &= \mathbb{L}^{2g-2} - \mathbb{L}^{2g}. \end{aligned}$$

<sup>17</sup>Since we can expand the zeta function in terms of symmetric powers.

<sup>18</sup>Note that  $\text{Conf}_{2g-2}(\mathbb{P}^1)$  has branch locus with degree  $2g - 2$ .