A ROUGH INTRO TO THE HERMITIAN K-THEORY SPECTRA KO AND KSp

Abstract. If/when you find any errors, send me an email at brazelton@math.harvard.edu

0. About

Notes from a talk at AMTRaK[1] on 2/28/25. The theme was Hermitian K-theory, despite the fact that I should be the last person people think of when looking for an authority on the subject. In an attempt to connect this theme with upcoming work of myself, Morgan Opie, and Tariq Syed, I wanted to give a rough overview of the ideas of Hermitian K-theory with the goal of defining KSp and KO as motivic spectra, and then discussing how we can leverage these to access information about stable and unstable motivic homotopy groups of spheres (as in [RSO19; AF14; ABH23] and similar work). This information is crucial in order to carry out explicit computations in motivic obstruction theory.

0.1. **References used.** Beyond the standard papers you'd expect, I consulted a few other references, including Fabien Hebestreit's 2020/2021 lecture notes on algebraic and Hermitian K-theory, Alexander Kupers' notes on Hermitian K-theory, and Arun Kumar's fantastic PhD thesis [Kum].

1. Why unimodular forms

We recall what *algebraic K-theory* is, at least in a classical sense – it is the process by which we take the category $\mathrm{Vect}(R)$ of algebraic vector bundles (finitely generated projective modules) over a ring R, consider its groupoid core $\mathrm{Vect}(R)^{\simeq}$, and then group complete:

$$K(R) := (\mathrm{Vect}(R)^{\simeq})^{\mathrm{gp}}.$$

Variants on K-theory come from trying to replicate this setup, with a little more structure added into the mix. An example this crowd might be familiar with is when R is a G-ring – in this case we can ask for a group completion that incorporates this action in some way, leading to *equivariant algebraic K-theory.*

Often in mathematics, a vector space or module comes handed to us with some kind of *form* on it. This could be a quadratic form, a symmetric bilinear form, a Hermitian form, a skew-symmetric form, etc. Let's recall some of these definitions from linear algebra.

Definition 1.1. We will say a *form* on a module M over a commutative base ring R to be any abelian group homomorphism

$$\beta \colon M \times M \to R.$$

We can ask for this to satisfy various properties, so we say that β is...

(1) ...*bilinear* if β is a morphism of R-modules
(2) ...*symmetric* if $\beta(v, w) = \beta(w, v)$ for all $v, w \in M$
(3) ...*skew-symmetric* if $\beta(v, w) = -\beta(w, v)$ for all $v, w \in M$

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(4) ...*alternating* if $\beta(v, v) = 0$ for all $v \in M$
(5) ...*even* if β takes values in $2R$

In the special case where $R = \mathbb{C}$ and M is some complex vector space, we can ask for the form to interact with complex conjugation in various ways. We then say β is...

(6) ...*sesquilinear* if $\beta(av, bw) = a\bar{b}\beta(v, w)$ for $a, b \in \mathbb{C}$ and $v, w \in M$
(7) ...*Hermitian* if β is sesquilinear and $\beta(v, w) = \overline{(\beta(w, v))}$
(8) ...*skew-Hermitian* if β is sesquilinear and $\beta(v, w) = -\overline{\beta(w, v)}$.

Example 1.2. The prototypical situation is when the base ring is $\mathbb{Z}$. In this case, a bilinear form is represented by a matrix over $\mathbb{Z}$. The columns of such a matrix span a *lattice* $\mathbb{Z}^n \subseteq \mathbb{R}^n$ which are topics of immense interest in mathematics.[2] We say a bilinear form over $\mathbb{Z}$ is *unimodular* if its determinant is ± 1.[3]

Example 1.3. If M is a real oriented manifold of dimension $2n$, then the cup product induces what's called its *intersection form*

$$H^n(M, \mathbb{Z})/\text{tors} \times H^n(M, \mathbb{Z})/\text{tors} \to H^{2n}(M, \mathbb{Z}) \cong \mathbb{Z}.$$

This is symmetric and bilinear, and often non-degenerate.

Definition 1.4. The *signature* of any symmetric bilinear form over $\mathbb{Z}$ (or any subring of $\mathbb{R}$) is defined by tensoring up to $\mathbb{R}$, diagonalizing so that only $-1, 0, 1$ appear on the diagonal, and then summing up these entries. The *signature* of an even-dimensional oriented manifold of its intersection form.

Definition 1.5. If β is a form over $\mathbb{Z}$ represented by some matrix $B \in M_n(\mathbb{Z})$, we say β is *even* if $v^T Bv \in 2\mathbb{Z}$ for every $v \in \mathbb{Z}^n$

Theorem 1.6 (Arf)**.** Any even unimodular lattice has signature divisible by 8.

Theorem 1.7 (Rokhlin)**.** If M is a smooth orientable closed 4-manifold with a spin structure ($w_2(M) = 0$) then its signature is divisible by 16.

Proof. $\pi_3\mathbb{S} = \mathbb{Z}/24$ □

In 4-manifold topology there is an incredibly close relationship between the intersection form and the manifold in question.

Theorem 1.8 (Freedman)**.** If Q is a unimodular symmetric bilinear form over $\mathbb{Z}$, there exists a simply connected closed 4-manifold M with Q as its intersection form. If Q is even, then M is uniquely determined up to homeomorphism.

For us here in algebraic topology, we're always interested in manifolds, so we're in particular curious about encoding data about the intersection form

$$H^n(M; R) \times H^n(M; R) \xrightarrow{\cup} H^{2n}(M; R) \cong R,$$

for whatever rings R and manifolds M we might be looking at. To that end, we'd like to take this story and approach it via a more categorical lens.

[2]These appear in sphere packing, Galois and Lie theory, Hodge theory, and so many other fields. In more applied math, lattice-based cryptography is a popular and well-studied area of research. This has been made even more popular by the fact that it appears to be the leading contender for post-quantum cryptographic protocols (at least according to NIST).

[3]Observe all changes of basis don't affect the determinant over $\mathbb{Z}$, since we can't scale by anything except ± 1, both of which square to 1.

2. Onwards to Hermitian K-theory

With a little motivation, we now might want to study modules over rings *equipped with* some kind of form, and we might want to group complete this thing, analogous to how we built algebraic K-theory.

An immediate issue we bump into is how to define the category we want to group complete. Do we want to try to define a "morphism of symmetric bilinear forms"? Ultimately we'll group complete so maybe we only define isomorphisms?

Naive attempt number 1: We might want to study all flavors of *forms* on R-modules M. Our naive idea might be saying that, since we're studying R-modules M, we might attempt to carry out this work in the category of R-modules. To see why this isn't general enough, recall the following:

Example 2.1. Recall that the ordinary inner product on $\mathbb{R}^n$, defined by

$$\langle x, y\rangle = \sum x_i y_i,$$

extends to an inner product on $\mathbb{C}^n$, given by

$$\langle w, z\rangle = \sum w_i \overline{z_i}.$$

This latter product is a map

$$\begin{aligned} \mathbb{C}^n \times \mathbb{C}^n &\to \mathbb{C} \\ (w, z) &\mapsto \langle w, z\rangle \end{aligned}$$

but this is *not* a map of complex vector spaces! This is precisely because of the appearance of complex conjugation. Instead, the inner product on $\mathbb{C}^n$ is a *Hermitian form.*

This isn't a choice we make, it's kind of forced on us. For the first part, we want norms $||v||$ to be real-valued (i.e. we want a reasonable notion of *length* for complex vectors), which forces an inner product on $\mathbb{C}^n$ not to be complex linear. Moreover, from our motivation of wanting ultimately to do something K*-theoretic* or *additive*, we should potentially be aware of the following:

Exercise 2.2. Given a sesquilinear form $\beta(-,-)$ on a complex vector space, write $v \perp_\beta w$ if $\beta(v, w) = 0$. In order to talk about orthogonal sums in a reasonable way, we would expect this relation to be symmetric, i.e. $v \perp_\beta w$ if and only if $w \perp_\beta v$. Verify that orthogonality with respect to a non-degenerate sesquilinear form β is a symmetric relation if and only if β is a scalar multiple of a Hermitian form.

So is there a category in which we can talk about alternating and symmetric forms over the real and complex numbers, together with Hermitian forms? The answer is yes, and the trick is to work in the world of $\mathbb{C} \otimes \mathbb{C}$-modules!

Notation 2.3. We want to consider $\mathbb{C}$ with two different $\mathbb{C} \otimes \mathbb{C}$-module structures, we'll denote these by $\mathbb{C}^{\mathrm{std}}$ and $\mathbb{C}^{\mathrm{Her}}$ for *standard* and *Hermitian* module structures.[4] We define them on pure tensors (then extend additively) in the following way:

$$\begin{aligned} \mathbb{C} \otimes \mathbb{C} \times \mathbb{C}^{\mathrm{std}} &\to \mathbb{C}^{\mathrm{std}} \\ (w_1 \otimes w_2, z) &\mapsto w_1 w_2 z, \end{aligned}$$

and

$$\begin{aligned} \mathbb{C} \otimes \mathbb{C} \times \mathbb{C}^{\mathrm{Her}} &\to \mathbb{C}^{\mathrm{Her}} \\ (w_1 \otimes w_2, z) &\mapsto w_1 \overline{w_2} z. \end{aligned}$$

[4] This is nonstandard notation and terminology, but I'd perhaps like to advocate for its use in this context?

We can then ask what it means to have a $\mathbb{C}\otimes\mathbb{C}$-module map $V\otimes V\to\mathbb{C}$ with either module structure, and we see we recover bilinear and sesquilinear forms.

Notation 2.4. For any complex vector space V, we then can define the sets of *symmetric bilinear* and *Hermitian* forms, respectively, as the C_2-equivariant maps

$$\begin{aligned}\mathrm{Bil}(V) &:= \text{bilinear forms on } V = \mathrm{Hom}_{\mathbb{C}\otimes\mathbb{C}}(V\otimes V, \mathbb{C}^{\mathrm{std}})\\ \mathrm{Sesq}(V) &:= \text{sesquilinear forms on } V = \mathrm{Hom}_{\mathbb{C}\otimes\mathbb{C}}(V\otimes V, \mathbb{C}^{\mathrm{Her}}).\end{aligned}$$

Let's zoom in to bilinear forms for a second. Note C_2 acts on $V\otimes V$ by swapping, and we can give $\mathbb{C}^{\mathrm{std}}$ a trivial C_2-action. Then if a form is C_2-equivariant, this means it is symmetric, which is fairly easy to see. Coinvariants require a bit more thought though. Recall:

Definition 2.5. A *quadratic form* $q\colon M\to R$ is a map so that $q(rm)=r^2q(m)$ and so that

$$q(m+n)-q(m)-q(n)$$

is bilinear.

Proposition 2.6. The map

$$\begin{aligned}\mathrm{Bil}(V)/\left\langle f-\bar{f}\right\rangle &\to \mathrm{Quad}_k(V)\\ g &\mapsto q(v)=g(v,v)\end{aligned}$$

is a bijection.

Proof. We first check the formula is well-defined, i.e. that $q(v)$ is indeed a quadratic form. We observe

$$q(rv)=f(rv,rv)=r^2f(v,v)=r^2q(v),$$

and that

$$\begin{aligned}q(v+w)-q(v)-q(w) &= f(v+w,v+w)-f(v,v)-f(w,w)\\ &= f(v,w)+f(w,v),\end{aligned}$$

which is bilinear in v and w since f is. We further see that the map is well-defined, since it factors through the quotient.

Moreover it is surjective, as any quadratic form is modeled by a bilinear form. To see it is injective, we must verify that if $g(v,v)=0$ for all v then g is symmetric. We first write

$$0=g(v+w,v+w)=g(v,v)+g(w,v)+g(w,w)+g(v,w)=g(w,v)+g(v,w).$$

So $g=-\bar{g}$ and $g=\bar{g}$, implying $2g=0$. If $2^{-1}\in k$, we're done. If $\mathrm{char}(k)=2$, we need a different argument, but I believe the proposition statement is still true.[5] □

Therefore we obtain

$$\begin{aligned}\mathrm{Sym}_{\mathbb{C}}(V) &:= \mathrm{Hom}_{\mathbb{C}\otimes\mathbb{C}}(V\otimes V,\mathbb{C}^{\mathrm{std}})^{C_2}\\ \mathrm{Quad}_{\mathbb{C}}(V) &= \mathrm{Hom}_{\mathbb{C}\otimes\mathbb{C}}(V\otimes V,\mathbb{C}^{\mathrm{std}})_{C_2}.\end{aligned}$$

There is always a norm map from the coinvariants to the invariants, and in this case it recovers the *polarization* of a quadratic form

$$\begin{aligned}\mathrm{Nm}\colon \mathrm{Quad}_{\mathbb{C}}(V) &\to \mathrm{Sym}_{\mathbb{C}}(V)\\ q &\mapsto \left[(v,w)\mapsto q(v+w)-q(v)-q(w)\right].\end{aligned}$$

[5] If anyone knows a reference, let me know!

This is because if q came from g, the norm map sends g to $g+\bar{g}$, and we see

$$(g+\bar{g})(v,w) = g(v,w)+g(w,v) = g(v+w,v+w)-g(v,v)-g(w,w) = q(v+w)-q(v)-q(2).$$

Note 2.7. We similarly get Hermitian forms in this way, working with $\mathbb{C}^{\mathrm{Her}}$ instead of $\mathbb{C}^{\mathrm{std}}$:

$$\mathrm{Her}(V) := \mathrm{Hom}_{\mathbb{C}\otimes\mathbb{C}}(V\otimes V, \mathbb{C}^{\mathrm{Her}})^{C_2}.$$

We can define Hermitian quadratic forms as coinvariants, and obtain a norm map in this context as well.

Note 2.8. If we are interested in *skew-symmetry*, we can give $\mathbb{C}^{\mathrm{std}}$ or $\mathbb{C}^{\mathrm{Her}}$ a non-trivial C_2-action, given by $z \mapsto -z$. Asking for C_2-invariants or coinvariants in this setting yields the study of skew-symmetricforms, or skew-Hermitian forms, respectively.

3. The general setup

Let's replace $\mathbb{C}$ with an arbitrary base ring R, and $\mathbb{C}^{\mathrm{std}}$ with any R-module M, considered with trivial C_2-action.[6] We replace V by any finitely generated projective R-module P. Then we define

$$\begin{aligned}
\mathrm{Sym}_R(P,M) &:= \mathrm{Hom}_{R\otimes R}(P\otimes P, M)^{C_2}\\
\mathrm{Quad}_R(P,M) &= \mathrm{Hom}_{R\otimes R}(P\otimes P, M)_{C_2}\\
\mathrm{Alt}_R(P,M) &= \{f \in \mathrm{Hom}_{R\otimes R}(P\otimes P, M)\colon f(p\otimes p)=0 \text{ for all } p\in P\}.
\end{aligned}$$

This gives symmetric forms, quadratic forms, and alternating forms (respectively) valued in M.

Remark 3.1.

(1) When M is R equipped with the standard $R\otimes R$-module structure coming from multiplication, we will drop the notation for the module in which forms are valued, and just write $\mathrm{Sym}_R(P)$, $\mathrm{Quad}_R(P)$, and $\mathrm{Alt}_R(P)$.
(2) If 2 is invertible, then being alternating is exactly the same as being skew-symmetric (see [Kum, Remark 2.2.1]). If 2 isn't invertible, we have to be a little more careful.
(3) There is always a norm map from coinvariants to invariants, and in the above it recovers the polarization of a quadratic form. We refer to things in its image as *even*:

$$\mathrm{Even}_R(P,M) = \mathrm{im}\left(\mathrm{Quad}_R(P,M) \xrightarrow{\mathrm{Nm}} \mathrm{Sym}_R(P,M)\right).$$

Again this subtlety only matters if 2 isn't invertible.

For any $f\in \mathrm{Hom}_{R\otimes R}(P\otimes P, M)$, we can restrict scalars along the map $R\cong \mathbb{Z}\otimes R\to R\otimes R$, and adjoint over to get a map from P to its "M-dual"

$$\mathrm{Hom}_{R\otimes R}(P\otimes P, M) \xrightarrow{\mathrm{res}^{R\otimes R}_{\mathbb{Z}\otimes R}} \mathrm{Hom}_R(P\otimes P, M) \cong \mathrm{Hom}_R(P, \mathrm{Hom}_R(P,M)).$$

Definition 3.2. A form $q\in \mathrm{Hom}_{R\otimes R}(P\otimes P, M)$ is called *unimodular* if the induced map

$$P \xrightarrow{\sim} \mathrm{Hom}_R(P,M)$$

is an equivalence.

[6] That's only for the purposes of this talk. In general we want M to be an $R\otimes R$-module, which is finite projective as an R-module, and $\sigma\colon M\to M$ a flip-linear involution, i.e. $\sigma^2=\mathrm{id}$ and $\sigma((x\otimes y)m)=(y\otimes x)\sigma(m)$.

As P varies, unimodular forms form a groupoid! We denote by $\mathrm{Unimod}^{\lambda}(R, M)$ the groupoid of all M-valued unimodular λ-forms over R, where λ is any of the adjectives we had before (symmetric, quadratic, even, skew-symmetric,...). We get natural maps

$$\mathrm{Unimod}^{q}(R, M) \to \mathrm{Unimod}^{e}(R, M) \to \mathrm{Unimod}^{s}(R, M).$$

Remark 3.3 (Important)**.** When 2 is invertible, these are equivalences of groupoids, and we will drop the superscript and write $\mathrm{Unimod}(R, M)$ to mean any of them.

We now have some stuff we can try to group complete in an ∞-categorical way, and the resulting objects will be different flavors of *Hermitian K-theory*. Before we do this, let's give one other classical perspective on this story.

4. Forms via categories with duality

The perspective we presented above is a specific case of a more general perspective, coming from the nine-author work on Hermitian K-theory and Poincaré ∞-categories [Cal+23a; Cal+23b; Cal+21]. This is the most general perspective, and it's what we should all probably learn. For our work we'll be content inverting two, so we can do away with a lot of these subtleties, and present the story in a slightly different fashion, which was the historical way in which Karoubi, Balmer, and others approached it.

Assumption: Let 2 always be invertible. Then quadratic form=symmetric bilinear form, alternating form=skew-symmetric form, etc.

Distilling the story for a minute, it seems that two things are particularly relevant to discuss forms over rings in this manner — one is the category $\mathrm{Vect}(R)$ of finitely generated projective modules, and the other is its *duality*. We recall that given an R-module P, a form $P \otimes P \to R$ is adjunct to a map $P \to \mathrm{Hom}_R(P, R)$, which is an isomorphism if and only if the form is non-degenerate. This module $\mathrm{Hom}_R(P, R)$ is the categorical dual of P if and only if P is finitely generated and projective.

To that end, we might try to extract the relevant data and generalize this to a more abstract categorical setting. This leads to the idea of *categories with duality*.

Definition 4.1 (see [Hor05, p. 1.1], cf. [Kum, p. 2.4.3])**.** A *category with (strict) duality* is a triple $(\mathscr{C}, D, \eta)$, where $D \colon \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ is a contravariant endofunctor, and $\eta \colon \mathrm{id}_{\mathscr{C}} \Rightarrow D^{\circ 2}$ is a natural isomorphism with the property that

$$\mathrm{id}_{DA} = D\eta_A \circ \eta_{DA}.$$

Definition 4.2 (see [Hor05, p. 1.2], cf. [Kum, p. 2.4.4])**.** If $\mathscr{C}$ is a category with duality, we denote by $\mathscr{C}_h$ its *Hermitian category*, whose objects are tuples of objects and isomorphisms $(x, x \xrightarrow{f} Dx)$ so that $f = f^*\eta$.[7] A morphism from (x, f) to (y, g) is a morphism $h \colon x \to y$ fitting into a commutative diagram

$$\begin{array}{ccc} x & \xrightarrow{\;h\;} & y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Dx & \xleftarrow[\;Dh\;]{} & Dy \end{array}$$

Definition 4.3. We define the Hermitian K-theory of a category with duality simply as

$$K^h(\mathscr{C}) := K(\mathscr{C}_h).$$

[7] todo **q**: What does this translate to in the module sense?

Why does this make sense? It turns out if $\mathscr{C}$ has some nice properties (it's the type of category you can apply K-theory to) then $\mathscr{C}_h$ does as well.

- If $\mathscr{C}$ is additive, then $(\mathscr{C}, \oplus)$ is symmetric monoidal under orthogonal sum [Hor05, p. 1.4]
- if $\mathscr{C}$ is exact then $\mathscr{C}_h$ is as well [Sch10].

Example 4.4. Let $\mathscr{C} = \mathrm{Mod}_R$ let $D_M := \mathrm{Hom}_R(-, M)$ for any R-module M, and let

$$\eta_M \colon \mathrm{id} \to \mathrm{Hom}_R(\mathrm{Hom}_R(-, M), M)$$

denote the double-dual for M. The sorts of modules for which we obtain a category with duality are precisely those for which η_M is a natural isomorphism.

(1) The Hermitian category attached to the ordinary triple of data $(\mathrm{Mod}(R), D_R, \eta_R)$ is exactly

$$(\mathrm{Mod}(R), D_R, \eta_R)_h = \mathrm{Unimod}(R).$$

(2) The Hermitian category attached to the triple with negative double dual $(\mathrm{Mod}(R), D_R, -\eta_R)$ is

$$(\mathrm{Mod}(R), D_R, -\eta_R)_h = \mathrm{Alt}(R)$$

(3) The Hermitian category attached to some modified duality, where M is an invertible rank one R-module is

$$(\mathrm{Mod}(R), D_M, \eta_M)_h = \mathrm{Unimod}(R, M).$$

Remark 4.5. Where did rings with involutions go? We should think about the component of the unit $\eta_R \colon R \to R$ on the base ring as encapsulating the data of the involution of our base ring, at least in the case of alternating forms as above.

Definition 4.6. Given a ring R, we define its *orthogonal K-theory* and its *symplectic K-theory* to be

$$\mathrm{KO}(R) := K^h(\mathrm{Unimod}(R))$$
$$\mathrm{KSp}(R) := K^h(\mathrm{Alt}(R)).$$

These deloop into spectra, letting us define negative Hermitian K-groups.

5. On plus constructions

Honest K-theory comes from looking at the groupoid $\mathrm{Vect}(R)^{\simeq}$ and seeing that it decomposes as $\amalg_{P \in \mathrm{Aut}_R(P)} B\mathrm{Aut}_R(P)$, then group completing this via a plus construction.

We might ask whether an analogous thing is true for the Hermitian K-theories we've built above. To do this we need to better understand what "automorphisms of a form" should mean, in both the symmetric and alternating contexts.

Definition 5.1. (1) If $q \in \mathrm{Unimod}(R)$ is a unimodular form over some $P \in \mathrm{Vect}(R)$, then we define its *orthogonal group* as

$$O(q) = \{f \in \mathrm{Aut}_R(P) \colon f^* q f = q\}.$$

This forms a group.[8]

(2) If $q \in \mathrm{Alt}(R)$ is an alternating/skew-symmetric form over some $P \in \mathrm{Vect}(R)$, we similarly define its *symplectic group* as

$$\mathrm{Sp}(q) := \{f \in \mathrm{Aut}_R(P) \colon f^* q f = q\}.$$

[8] It's also a group scheme, but we don't use that structure until we head into motivic land.

Just as we have the "trivial" or "free" case of $P = R^{\oplus n}$ where we recover the general linear group as $\mathrm{Aut}_R(R^{\oplus n})$ in $\mathrm{Mod}(R)$, we have some free cases here as well:

Example 5.2. We use $O(n)$ or O_n to refer to the trivial quadratic form on R^n. We denote by Sp_{2n} the symplectic group associated to the alternating form Ω_n on R^{2n} defined by the matrix

$$\Omega_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Notation 5.3. We denote by

$$O := \mathrm{colim}_n O_n$$
$$\mathrm{Sp} := \mathrm{colim}_n \mathrm{Sp}_n.$$

These colimits can be in discrete groups or group schemes.

We then get an analogous result to Quillen's plus construction, due to Karoubi.

Theorem 5.4 ([Kar73])**.** Let R be a regular ring with 2 inverted. Then we have equivalences of E_∞-groups

$$\mathrm{KO}(A) \cong \mathrm{GW}(A) \times \mathrm{BO}(A)^+$$
$$\mathrm{KSp}(A) \cong \mathrm{KSp}_0(A) \times \mathrm{BSp}(A)^+$$

Again there's a more general notion when R has involution (see e.g. the discussion in the introduction of [HS04]).

Notation 5.5. Analogous to ordinary algebraic K-theory we denote by $\mathrm{KO}_n := \pi_n \mathrm{KO}$ and similarly for KSp.

Remark 5.6. The E_∞-groups $\mathrm{KO}(A)$ and $\mathrm{KSp}(A)$ are infinite loop spaces as constructed above. They can be delooped, and one can define negative KO and KSp-groups this way.

Remark 5.7. There are always forgetful functors

$$\mathrm{Unimod}(R) \to \mathrm{Vect}(R),$$
$$\mathrm{Alt}(R) \to \mathrm{Vect}(R)$$

which induce forgetful maps to algebraic K-theory. Similarly we can send any projective module P to $P \oplus P$ equipped with the alternating form Ω_n or the hyperbolic form. These give what are called *hyperbolic maps*

$$K(R) \to \mathrm{KO}(R)$$
$$K(R) \to \mathrm{KSp}(R).$$

Asking what the composites of hyperbolic and forgetful are, in either order, is a really natural and interesting question.

6. Promoting to motivic spectra

We now have an assignment (again working over $\mathbb{Z}[1/2]$):

$$\mathrm{CRing} = \mathrm{Aff}^{\mathrm{op}} \to \mathcal{S}$$
$$A \mapsto \mathrm{KO}(A)$$
$$A \mapsto \mathrm{KSp}(A).$$

We can ask whether these form motivic spectra – in other words,

- are they *Nisnevich sheaves*?

- are they $\mathbb{A}^1$-*invariant*, i.e. are they sensitive to plugging in $A[t]$ instead of A?
- are they $\mathbb{P}^1$-*spectra*, meaning do they have some kind of periodicity?

Proposition 6.1 ([Hor05, p. 1.14])**.** KO and KSp are Nisnevich excisive on regular affine schemes.

Proof sketch. If we have a square of rings (Nisnevich distinguished after Spec) of the form[9]

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_f & \longrightarrow & B_{\phi(f)}, \end{array}$$

then we obtain a cartesian square of stable ∞-categories

$$\begin{array}{ccc} \mathrm{Perf}(A) & \longrightarrow & \mathrm{Perf}(B) \\ \downarrow & \lrcorner & \downarrow \\ \mathrm{Perf}(A_f) & \longrightarrow & \mathrm{Perf}(B_{\phi(f)}). \end{array}$$

Since K is an additive invariant, the proof suffices to verifying that the above square is still Cartesian on the associated Hermitian categories, with either symmetric or symplectic structures. ☐

Proposition 6.2 (Karoubi, Balmer, [Hor05, p. 1.12])**.** If R is a regular ring, we have homotopy equivalences

$$\mathrm{KO}(R) \xrightarrow{\sim} \mathrm{KO}(R[t])$$
$$\mathrm{KSp}(R) \xrightarrow{\sim} \mathrm{KSp}(R[t].$$

Sketch. For KO, this follows from Quillen's fundamental theorem of algebraic K-theory, together with some analogue of *Harder's theorem*, which says that quadratic forms are constant in a family, at least over a field. The Harder-flavored result we're referring to is [Bal01, p. 3.1]. For KSp, this is somewhere in Karoubi, at least that's the reference Matthias gives in MO197767. ☐

This still doesn't tell us how to define Hermitian K-theory on arbitrary schemes, only affine ones. We can use the Jouanalou device though!

Definition 6.3 ([Hor05, p. 2.2])**.** For X any regular scheme, we define $\mathrm{KO}(X) := K^h(W)$ for $W \to X$ any affine bundle torsor.

Hornbostel mentions this is well defined by the appendix and A.2 in Weibel's KH paper.

Corollary 6.4. We obtain a motivic infinite loop space $\mathrm{KO} \in \mathrm{SH}(R)$ for every regular base ring R, (really every regular qcqs base scheme) in which 2 is invertible.

7. Periodicity and $\mathbb{P}^1$-spectra

We can think about $\mathrm{KO}(X)$ as $\mathrm{KO}(\mathrm{Perf}(X))$. To that end, we might consider bilinear or symplectic forms valued in $\mathscr{O}_X$, concentrated in degree zero. However we might, alternatively, consider forms valued in $\mathscr{O}_X[k]$ for some $k \in \mathbb{Z}$. We denote by $\mathrm{KO}^{[k]}(X)$ this Hermitian K-theory with shifted duality.

[9] Here $A \to B$ is étale, $f \in A$ and the induced map $A/f \xrightarrow{\sim} B/\phi(f)$ is a ring homomorphism

There's a notion of *relative* groups $\mathrm{KO}(X, U)$ for $U \subseteq X$ open, which is due to Schlichting. We will black box this, but note that by a dévissage argument, we can obtain weak equivalences

$$\mathrm{KO}^{[k]}(X) \xrightarrow{\sim} \mathrm{KO}^{[k+1]}(X \times \mathbb{A}^1, X \times \mathbb{G}_m),$$

which are adjoint to maps

$$\mathrm{KO}^{[k]}(X) \times \mathbb{P}^1 \to \mathrm{KO}^{[k+1]}(X).$$

Proposition 7.1. These form a $\mathbb{P}^1$-spectrum $\mathrm{KO} \in \mathrm{SH}(S)$.

Additionally, any $(V, \phi) \in \mathrm{KO}(X)$ gives rise to a nondegenerate symmetric bilinear form

$$V[n] \otimes_{\mathscr{O}_X} V[n] \to \mathscr{O}_X[4n],$$

inducing a map $\mathrm{KO}(X) \to \mathrm{KO}^{[4n]}(X)$ for any n.[10] Similarly a *symplectic form* (E, ϕ) induces a symmetric bilinear form

$$E[2n+1] \otimes_{\mathscr{O}_X} E[2n+1] \to \mathscr{O}_X[4n+2].$$

See for instance [PW18, p.6].

Theorem 7.2 (Schlichting)**.** These induced maps

$$\mathrm{KO}^{[n]}(X) \to \mathrm{KO}^{[4n]}(X)$$
$$\mathrm{KSp}(X) \to \mathrm{KO}^{[4n+2]}(X)$$

are equivalences.

Corollary 7.3. Hermitian and symplectic K-theory are really periodic parts of *one motivic spectrum*, which is 4-fold periodic with respect to $\mathbb{P}^1$.

On complex points, the homotopy type of the fourfold smash product of $\mathbb{P}^1$ is S^8, hence we should think of this as analogous to ordinary Bott periodicity.

8. Accessing motivic homotopy groups of spheres

Let's further simplify and work over a field k in which 2 is invertible. As a ring spectrum, $\mathrm{KO} \in \mathrm{SH}(k)$ receives a unit map

$$\mathbb{S} \to \mathrm{KO}.$$

This is an isomorphism on $\pi_0^{\mathbb{A}^1}$. Working with KO instead of the sphere spectrum has several advantages, including that KO is SL^c-oriented, so we can leverage this map to get $\mathrm{GW}(k)$-valued Euler classes. Another advantage is it helps us access the 1-stem.

Proposition 8.1. The unit map $\mathbb{S} \to \mathrm{KO}$ factors through its 1-effective cover $f_0\mathrm{KO}$.

Theorem 8.2 ([RSO19])**.** There is a short exact sequence

$$0 \to \mathbf{K}^{\mathrm{M}}_{2-n}/24 \to \pi_{n+1,1}\mathbb{S} \to \pi_{n+1,n} f_0\mathrm{KO}.$$

The rightmost map is surjective for $n \geq -4$.

[10] If we tried to do this for n instead of $2n$, the resulting form wouldn't be symmetric.

This is the stable story, and there is also an unstable one. We have a filtration on Sp_∞ by the even Sp_{2n}'s, giving rise to fiber sequences

$$\mathrm{Sp}_{2n} \to \mathrm{Sp}_{2n+2} \to \mathbb{A}^{2n} \setminus 0$$

for each n, and these assemble into an exact couple spectral sequence

$$E_1^{s,t} = \pi_t^{\mathbb{A}^1}(\mathbb{A}^{2s} \setminus 0) \Rightarrow \pi_t^{\mathbb{A}^1}\mathrm{Sp} = \pi_{t+1}^{\mathbb{A}^1}\mathrm{BSp} = \mathbf{K}_{t+1}^{\mathrm{Sp}}.$$

The edge maps here go from unstable homotopy groups of spheres to Hermitian K-theory, and they are some unstable analogue of the unit map $\mathbb{S} \to \mathrm{KSp}$. These allow us access to short exact sequences of sheaves.

Theorem 8.3 ([AF14, Thm. 3, Cor. 4.9])**.** We have a short exact sequence of sheaves

$$0 \to \mathbf{T}_4' \to \pi_2^{\mathbb{A}^1}(\mathbb{A}^2 \setminus 0) \to \mathbf{K}_3^{\mathrm{Sp}} \to 0,$$

where $\mathbf{T}_4'$ is related to mod 12 Milnor K_4^M.[11]

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[11] There is an exact sequence

$$\mathbf{I}^g \to \mathbf{T}_4' \to \mathbf{S}_4' \to 0,$$

and $\mathbf{K}_4^M/12 \twoheadrightarrow \mathbf{S}_4'$ an iso after 2fold contraction.

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