K-THEORY OF INFINITY CATEGORIES

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Abstract. Notes from an expository talk given in the algebraic K-theory seminar at UPenn, spring 2020.

0.1. References.

- Barwick, *On the Algebraic K-Theory of Higher Categories*
- Blumberg, Gepner, Tabuada (BGT), *A Universal Characterization of Higher Algebraic K-Theory*
- Brasca, K-theory of Waldhausen categories
- Lurie, MATH281, Lectures 14, 16
- Lurie, *Higher Topos Theory*
- Lurie, *Kerodon*
- Joyal, *The Theory of Quasi-Categories and its Applications*

1. Infinity Categories

1.1. **Intuition.** Let `Cat` be the category of all small categories. This had the following data:

small categories	objects
functors	morphisms
natural transformations	morphisms between morphisms.

If we consider natural transformations as part of the data of `Cat`, we have considerably more data than an ordinary category (often called a 1-category). Here we call `Cat` a *2-category*, and we can call the natural transformations *2-morphisms*.

1.2. **Enrichment.** The reason we were able to reasonably talk about 2-morphisms in `Cat` is due to the following observation:

For any $\mathscr{C}, \mathscr{D} \in$ `Cat`, we have that $\mathrm{Fun}(\mathscr{C}, \mathscr{D})$ is a 1-category, whose objects are functors and whose morphisms are natural transformations.

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This leads to the following ad hoc definition: a *2-category* is any category whose homs are 1-categories.

To make this more explicit we would need the notion of enriched categories, which we won't discuss here. However to soup this up to a legitimate definition, we would say a *2-category* is any category enriched in a category of 1-categories.

1.3. **Infinity categories.** Inductively, we think about *n-categories* as being any category enriched in a category of $(n-1)$-categories. That means homs in an n-category are $(n-1)$-categories, and we think about n-morphisms as being morphisms between $(n-1)$-morphisms.

If we have n-morphisms for every n, then we say that we have an *∞-category.* We remark though that we can always view any category as an ∞-category by just letting all the higher morphisms be the identity.

An (n, r)-category is a category for which all k-morphisms with $k > n$ are trivial, and all k-morphisms with $k > r$ are equivalences (will come back to a definition of this).

We could inductively define an $(n + 1, r + 1)$-category to be any category enriched in a category of (n, r)-categories. In particular an $(\infty, 1)$-category is any category enriched in a category of $(\infty, 0)$-categories.

Examples 1.1.

- a $(1, 1)$-category is an ordinary category
- a $(1, 0)$-category is a groupoid
- an "∞-category" generally refers to an $(\infty, 1)$-category, that is a category with higher morphisms above degree n invertible.

1.4. **Spaces are ∞-groupoids.**

Definition 1.2. An *∞-groupoid* is an $(\infty, 0)$-category.

For example, any topological space canonically determines an $(\infty, 0)$-groupoid as follows:

0-morphisms (objects)	points
1-morphisms	directed paths between points
2-morphisms	homotopies of paths
3-morphisms	homotopies of homotopies
$\vdots$	$\vdots$

1.5. **Various models.** Referring to something as an "infinity-category" is a bit loaded. In order to get a good handle on infinity categories, we should have some type of *model* of what a good theory of infinity categories should look like.

In particular a model should be a home for infinity categories, i.e. *a category whose objects are infinity categories.* In our previous example, for instance, we saw that `Top` was a good model for ∞-groupoids.

Here are a few models of $(\infty, 1)$-categories:

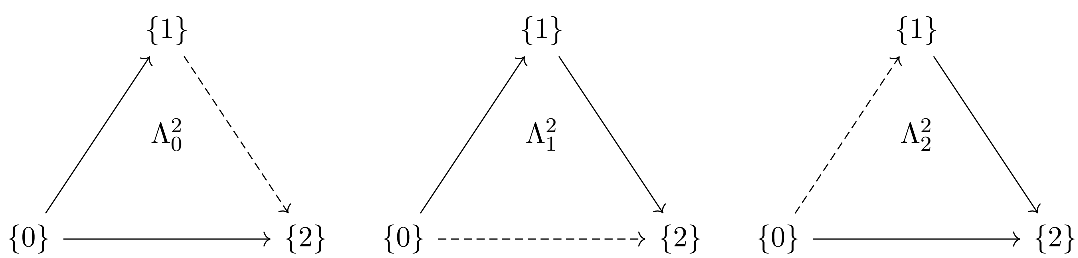
- quasi-categories
- simplicially enriched categories
- topologically enriched categories
- Segal categories
- complete Segal spaces

These are all "equivalent" in the sense that they have model structures and Quillen equivalences between them.

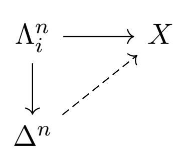
For the remainder of this talk, an $(\infty, 1)$-category will mean a quasi-category.

1.6. **Quasi-categories.** Recall that we have a standard n-simplex $\Delta^n \in$ `sSet`. Its boundary is denoted by $\partial\Delta^n$.

Definition 1.3. The *ith horn*, denoted Λ_i^n, is the boundary $\partial\Delta^n$ minus the face opposite the ith vertex.

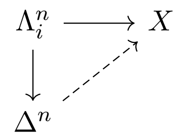


Definition 1.4. We say that a simplicial set X is a *quasicategory* if any inclusion of a horn Λ_i^n, with $0 < i < n$, extends to an inclusion of the n-simplex:

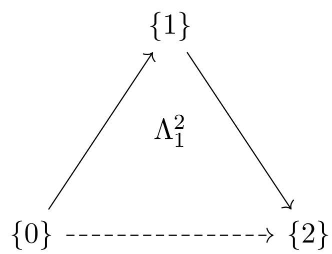


We denote by `qCat` the full subcategory of `sSet` containing all quasi-categories.

1.7. **Horn filling.** Morally, what does it mean for the dashed arrow to exist:



Consider the smallest example: Λ_1^2.



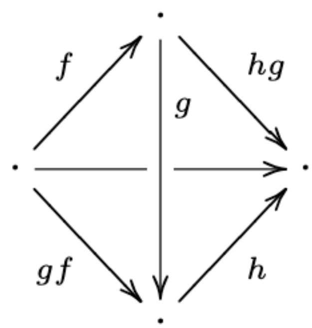
The inclusion of this into the simplicial set X corresponds to the selection of three 0-cells (which we consider to be objects) and two composable 1-cells (which we consider to be morphisms). The existence of a "filling" of this horn (an extension to the 2-simplex) means that you can compose these 1-cells in a way that *maybe doesn't commute strictly*, but commutes up to some 2-cell.

Analogously, filling a horn Λ_i^n for $0 < i < n$ means that for any composable collection of $(n-1)$-morphisms in a quasi-category X, there is a way to compose them weakly in X, where the composition is witnessed by some n-cell.

1.8. **Nerves of categories: Λ_1^2.** Suppose $X = N(\mathscr{C})$ is the simplicial set obtained as the nerve of some small category $\mathscr{C}$ (remember the nerve had as n-cells strings of n-composable morphisms in $\mathscr{C}$).

For any inclusion $\Lambda_1^2 \to N\mathscr{C}$, this specifies two morphisms f and g in $\mathscr{C}$ which are composable. We remark that this horn can be filled uniquely by the composite $g \circ f$, and the 2-cell witnessing this composition is the identity.

1.9. **Nerves of categories: Λ_2^3.** The image of $\Lambda_2^3 \to N\mathscr{C}$ looks like:



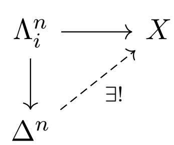
where the back face is missing. The bottom face exists, so the back unlabelled arrow must be equal to the composite of h and gf, giving the arrow $h \circ (gf)$. In order to fill the back face, we must see that the back arrow commutes up to some higher cell, that is, there is a 2-cell witnessing the composite $(hg) \circ f \Rightarrow h \circ (gf)$.

However *because $\mathscr{C}$ was a category*, we have associativity of morphisms. Thus the back face fills, and the entire 3-cell in the center fills, corresponding to the fact that the following composites are equal:

$$h \circ g \circ f = h \circ (g \circ f) = (h \circ g) \circ f.$$

1.10. **Nerves of categories: higher horns.** As you might imagine, filling other horns in $N\mathscr{C}$ has analogous interpretations, corresponding to various ways to compose n-composable morphisms. Moreover since the composition is strict (higher cells witnessing this composition are the identity) we have the following result.

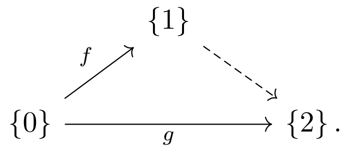
Proposition 1.5. The nerve of any small 1-category is a quasi-category; moreover, horns fill uniquely: for $0 < i < n$ we have a unique dashed map



This condition is actually sufficient to recognize when a simplicial set arises as the nerve of a category.

Proposition 1.6. We have that a simplicial set X is the nerve of a category $\mathscr{C}$ if and only if for all $0 < i < n$, the inclusion of any horn $\Lambda^n_i \to X$ extends *uniquely* to the inclusion of an n-simplex.

1.11. **Outer horns.** For quasi-categories, we said that we wanted filling for horns Λ^n_i where $0 < i < n$. Why shouldn't we expect filling for $i = 0, n$? Consider the following example:



Say we were mapping $\Lambda^2_0 \to N\mathscr{C}$ to the nerve of a category. In order for the dashed map to exist, it must be equal to gf^{-1}, that is, f must be an isomorphism in the category $\mathscr{C}$. In general there is no way to guarantee this. However if all maps in $\mathscr{C}$ were isomorphisms, then we would have this filling.

Proposition 1.7. The nerve of a groupoid admits horn filling for all Λ^n_i, where $0 \leq i \leq n$.

Definition 1.8. If a simplicial set X admits horn filling for all Λ^n_i for $0 \leq i \leq n$, we say it is a *Kan complex*.

The category $\texttt{Kan}$ of Kan complexes serves as a model for $(\infty, 0)$-categories.

1.12. **Hom-sets.** The full subcategory $\texttt{qCat} \subseteq \texttt{sSet}$ serves as a model for $(\infty, 1)$-categories. In particular for $C, D \in \texttt{qCat}$, we define an ∞-*functor* $F : C \to D$ to just be any morphism in the ambient category of simplicial sets.

Let $X \in \texttt{qCat}$, then for two vertices $a, b \in X_0$ (remember these are supposed to be objects) we should describe $\mathrm{Hom}_X(a, b) =: X(a, b)$. By our discussion of enrichment, we should expect this object to be a Kan complex.

Consider the source and target maps

$$(s, t) : \mathrm{Hom}_{\texttt{qCat}}(\Delta^1, X) \to \mathrm{Hom}_{\texttt{qCat}}(\Delta^0 \amalg \Delta^0, X) = X \times X,$$

and let $X(a, b)$ denote the fiber of this map over the pair (a, b). We define this to be the hom-object $\mathrm{Hom}_X(a, b)$.

Proposition 1.9. [Lur09, p. 1.2.2.3] If $X \in \texttt{qCat}$ then $X(a, b) \in \texttt{Kan}$ for any $a, b \in X_0$.

1.13. **Adjunctions.** Let $\mathscr{C}$ and $\mathscr{D}$ be quasi-categories, and let $a \in \mathscr{C}$ and $b \in \mathscr{D}$. We say that ∞-functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are *adjoint* if we have a natural weak equivalence of Kan complexes

$$\mathrm{Hom}_{\mathscr{D}}(Fa, b) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{C}}(a, Gb).$$

Here a weak equivalence of Kan complexes means a weak equivalence after geometric realization.

1.14. **Terminal objects.** Let $\mathscr{C} \in \texttt{qCat}$.

Definition 1.10. We say that $x \in \mathscr{C}_0$ is *terminal* if, for every $a \in \mathscr{C}_0$, the Kan complex $\mathscr{C}(a, x)$ is contractible (meaning its geometric realization is contractible). Similarly, $x \in \mathscr{C}_0$ is *initial* if $\mathscr{C}(x, a)$ is contractible for all $a \in \mathscr{C}_0$.

We say $\mathscr{C}$ is *pointed* if it has a *zero object*, which is an object that is both initial and terminal.

In general limits and colimits are hard to construct, see Higher Topos Theory Chapter 4 for more detail.

1.15. **Cofibers.** Suppose that $\mathscr{C}$ is a quasi-category which has a zero object, denoted $*$, and pushouts.

Definition 1.11. The *cofiber* of a morphism $f : X \to Y$ in $\mathscr{C}$ is define to be the pushout

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle !}\downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \mathrm{cofib}(f) \end{array}$$

We refer to a sequence $X \xrightarrow{f} Y \to \mathrm{cofib}(f)$ as a *cofiber sequence*.

Definition 1.12. The *suspension* of X is defined to be the cofiber of the unique map $X \xrightarrow{!} *$:

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & \ulcorner & \downarrow \\ * & \longrightarrow & \Sigma X. \end{array}$$

2. K_0 for infinity categories

2.1. $K_0(\mathscr{C})$**.** Let $\mathscr{C}$ be a pointed ∞-category admitting pushouts. Then define $K_0(\mathscr{C})$ to be the free abelian group $[X]$ on objects of $\mathscr{C}$ modulo that a cofiber sequence

$$Z \to X \to Y$$

gives the relation $[Z] + [Y] = [X]$.

Exercise 2.1. $K_0(\mathscr{C})$ is abelian.

Exercise 2.2. We have that $[*] = 0$.

Exercise 2.3. We have that $[\Sigma X] = -[X]$.

Warning: If $\mathscr{C}$ admits infinite coproducts, then any object X fits into a cofiber sequence

$$\amalg_{n\geq 1} X \to \amalg_{n\geq 0} X \to X,$$

for which we see $[X] = 0$.

2.2. **Functoriality of $K_0(\mathscr{C})$.** Suppose $\mathscr{C}$ and $\mathscr{D}$ are pointed ∞-categories with pushouts. What conditions do we need on a functor $F : \mathscr{C} \to \mathscr{D}$ to induce a group homomorphism $K_0(\mathscr{C}) \to K_0(\mathscr{D})$?

Clearly we need F to preserve the zero object. Moreover we need that $[F(X \amalg Y)] = [F(X)] + F[Y]$, that is, since $X \to X \amalg Y \to Y$ is a cofiber diagram, so must be $F(X) \to F(X \amalg Y) \to F(Y)$. Therefore we should require F to preserve cofiber sequences as well.

Example 2.4. We remark that Σ was a colimit itself, thus it preserves all finite colimits. The functor $\Sigma : \mathscr{C} \to \mathscr{C}$ induces the multiplication by (-1) map on $K_0(\mathscr{C})$.

2.3. **Stable ∞-categories.**

Definition 2.5. We say an ∞-category $\mathscr{C}$ is *stable* if it is pointed, has pushouts, and so that the endofunctor $\Sigma : \mathscr{C} \to \mathscr{C}$ is an equivalence of categories.

Properties of stable ∞-categories:

(1) $\mathscr{C}$ has all finite limits and colimits
(2) A square in $\mathscr{C}$ is a pullback square if and only it is a pushout square
(3) A functor between stable ∞-categories preserves the zero object and cofibers if and only if it preserves all finite colimits.

Example 2.6. The ∞-category of spectra is stable.

For any category, it admits a *stabilization*, that is a functor to a stable infinity category, initial among such functors. This is given by the *Spanier-Whitehead category* $\mathrm{SW}(\mathscr{C})$, defined as the colimit:

$$\mathrm{SW}(\mathscr{C}) := \operatorname{colim}\left(\mathscr{C} \xrightarrow{\Sigma} \mathscr{C} \xrightarrow{\mathscr{C}} \xrightarrow{\Sigma} \cdots\right).$$

Remark 2.7. We see that Σ preserves all colimits, therefore

$$K_0(\mathscr{C}) \simeq K_0(\mathrm{SW}(\mathscr{C})).$$

So without loss of generality, for K_0, we can assume we are working with stable ∞-categories (this will be true in general).

3. Constructing higher K-theory

3.1. **Reminder: K-theory of a Waldhausen category.** Briefly, we had a category $\mathscr{C}$ with cofibrations and weak equivalences.

- we built categories $S_n\mathscr{C}$, whose objects were these "inverted staircase" diagrams of pushouts
- this gave a bisimplicial set $S_\bullet\mathscr{C}$, for which we could take any type of geometric realization, which all yielded equivalent spaces
- the fundamental group of this space was $K_0(\mathscr{C})$, which is shifted from what we want, so we loop the space to define $K(\mathscr{C})$.

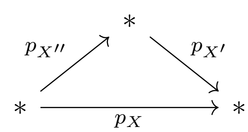
3.2. **Waldhausen K-theory of ∞-categories. Goal**: to replicate the construction of Waldhausen K-theory for ∞-categories in order to define the higher K-theory of ∞-categories.

This will proceed as follows:

- starting with an ∞-category $\mathscr{C}$, we get the abelian group $K_0(\mathscr{C})$
- build categories $S_n\mathscr{C}$, which under the nerve functor are considered as ∞-categories
- take the geometric realization of this bisimplicial set
- again, take the loop space to arrive at $K(\mathscr{C})$

3.3. **Objects as paths: 2-simplices.** As in the Waldhausen construction for Waldhausen categories, we want to build a based space W, where each $[X] \in K_0(\mathscr{C})$ corresponds to a path p_X in W beginning and ending at the base point $*$.

For a cofiber sequence $X' \to X \to X''$ we want the paths $p_{X'} \circ p_{X''}$ and p_X to be homotopic, in order to encode the relations on $K_0(\mathscr{C})$ as relations in $\pi_1(W)$. That is, we need a 2-simplex:



3.4. **3-simplices.** What can we say for an arbitrary pair of maps $X \xrightarrow{f} Y$ and $Y \xrightarrow{g} Z$, not necessarily forming a cofiber diagram?

Proposition 3.1. We have that

$$[Z] = [X] + [Y/X] + [Z/Y].$$

Proof 1. Use the cofiber diagrams

$$\begin{aligned} X \to Z \to Z/X \quad &\rightsquigarrow \quad [Z] = [X] + [Z/X] \\ (Y/X) \to (Z/X) \to (Z/Y) \quad &\rightsquigarrow \quad [Z/X] = [Y/X] + [Z/Y]. \end{aligned}$$

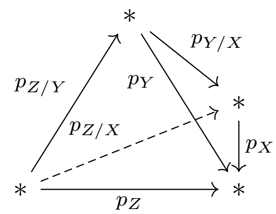
□

Proof 2. Use the cofiber diagrams

$$\begin{aligned} Y \to Z \to Z/Y \quad &\rightsquigarrow \quad [Z] = [Y] + [Z/Y] \\ X \to Y \to Y/X \quad &\rightsquigarrow \quad [Y] = [X] + [Y/X]. \end{aligned}$$

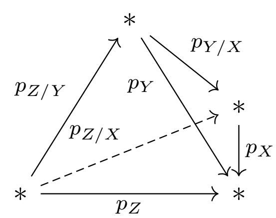
□

We can compile all of this into the following 3-simplex:



Analogous information is available for any string of composable morphisms — how do we encode this information simplicially?

3.5. **3-simplices, continued.**



We could also encode this information via the following diagram, where we stipulate that all rectangles in sight are pushout diagrams:

$$\begin{array}{ccccccc} * & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ && \downarrow && \downarrow && \downarrow \\ && * & \longrightarrow & Y/X & \longrightarrow & Z/X \\ &&&& \downarrow && \downarrow \\ &&&& * & \longrightarrow & Z/Y \\ &&&&&& \downarrow \\ &&&&&& * \end{array}$$

When looking for higher analogs for how to encode the cofiber relations induced by a composite $X_1 \to X_2 \to \cdots \to X_n$ of maps, we obtain the correct notion by generalizing this diagram above.

3.6. **Higher simplices.** Thus when building our space W, we should adjoin an n-simplex for every diagram of the form

$$\begin{array}{ccccccc}
* & \longrightarrow & X_1 & \longrightarrow & \cdots & \longrightarrow & X_n \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & * & \longrightarrow & \cdots & \longrightarrow & X_n/X_1 \\
 & & & & \downarrow & & \downarrow \\
 & & & & \cdots & \longrightarrow & \cdots \\
 & & & & & & \downarrow \\
 & & & & & & *
\end{array}$$

Let's formalize this— let $[n]$ be the ordered set $\{0 < 1 < \cdots < n\}$, and let

$$[n]^{(2)} := \{(i,j) \in [n] \times [n] \ : \ i \leq j\}.$$

Then, by associating $[n]^{(2)}$ with its nerve, which is an ∞-category, we should view our n-simplices as objects of the ∞-functor category

$$\mathrm{Fun}(N([n]^{(2)}), \mathscr{C}).$$

3.7. **Gapped objects.** Explicitly we define an $[n]$-*gapped object of* $\mathscr{C}$ to be a functor $X : N([n]^{(2)}) \to \mathscr{C}$ so that

(1) for each $i \in [n]$ we have that $X(i,i) \cong *$ in $\mathscr{C}$ is the zero object
(2) for each $i \leq j \leq k$ we have a pushout diagram

$$\begin{array}{ccc}
X(i,j) & \longrightarrow & X(i,k) \\
\downarrow & & \downarrow \\
X(j,j) & \longrightarrow & X(j,k),
\end{array}$$

equivalently using the previous condition, we have a cofiber sequence

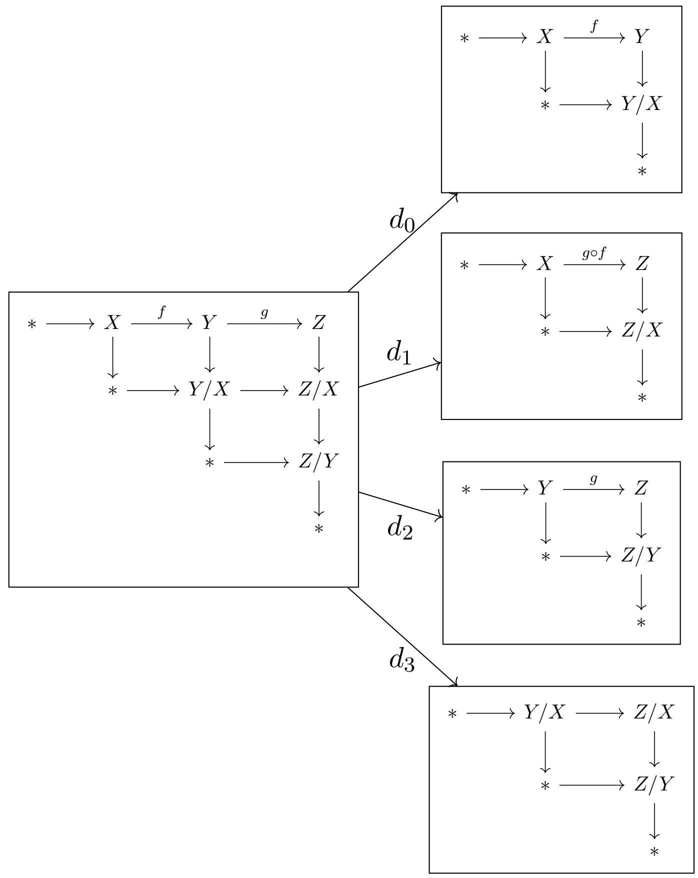
$$X(i,j) \to X(i,k) \to X(j,k).$$

We denote by $\mathrm{Gap}_{[n]}(\mathscr{C})$ the collection of all $[n]$-gapped objects. This forms an ∞-category.

Proposition 3.2. The inclusion functor `Kan` $\to$ `qCat` admits a right adjoint, which provides the largest Kan complex contained in a quasicategory.

We denote by $S_n(\mathscr{C})$ the largest Kan complex contained in $\mathrm{Gap}_{[n]}(\mathscr{C})$.

3.8. **Face and degeneracy maps.** For $S_2(\mathscr{C}) \subseteq \mathrm{Gap}_{[n]}(\mathscr{C})$, we provide the face maps:



3.9. **Degeneracy maps.** Degeneracy maps are less interesting, we simply add in an identity along the top row and add in identities horizontally going down.

3.10. **The simplicial Kan complex.** We now have a simplicial Kan complex $S_\bullet \mathscr{C}$. Regarding this as a bisimplicial set, we can take its geometric realization. We then define the *K-theory space*:

$$K(\mathscr{C}) := \Omega \left| S_\bullet \mathscr{C} \right|.$$

Properties:

- if $F : \mathscr{C} \to \mathscr{D}$ preserves finite colimits, it induces a continuous map $K(\mathscr{C}) \to K(\mathscr{D})$
- the projection functors $\mathscr{C} \times \mathscr{D} \to \mathscr{C}$ and $\mathscr{C} \times \mathscr{D} \to \mathscr{D}$ preserve finite colimits. These maps induce a homotopy equivalence

$$K(\mathscr{C} \times \mathscr{D}) \xrightarrow{\sim} K(\mathscr{C}) \times K(\mathscr{D})$$

- the coproduct functor

$$\amalg : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$$
$$(X,Y) \mapsto X \amalg Y$$

  preserves colimits, and therefore ascends to a monoidal structure

$$K(\mathscr{C}) \times K(\mathscr{C}) \to K(\mathscr{C}).$$

  This turns $K(\mathscr{C})$ into a grouplike E_∞-space, that is, an infinite loop space.
- The map $\mathscr{C} \to \mathrm{SW}(\mathscr{C})$ induces an equivalence

$$K(\mathscr{C}) \xrightarrow{\sim} K(\mathrm{SW}(\mathscr{C})).$$

3.11. **Examples.**

(1) If R is a ring, we can take $D^b(R)$, the derived category of the ring, which can be viewed as a stable ∞-category. We have that

$$K(D^b(R)) \cong K(R) \cong \mathrm{BGL}(R)^+,$$

therefore we recover the algebraic K-theory of the ring.

(2) Given a scheme X, we can take its category $\mathrm{Perf}(X)$ of perfect complexes, which has the structure of a stable ∞-category. Taking its K-theory we recover Thomason-Trobaugh K-theory

(3) Given a topological space X, its singular chains $\mathrm{Sing}(X)$ is a Kan complex, and therefore an ∞-category. Let $\mathscr{C} \subseteq \mathrm{Fun}(\mathrm{Sing}(X), \mathtt{Sp})$ be the subcategory on *compact objects* (see nLab). Then $K(\mathscr{C}) \simeq A(X)$ is the *A-theory* of the space X.

4. Additivity

4.1. **Additivity theorem (classically).** Let $\mathcal{E}(\mathscr{C})$ be the category whose objects are exact sequences $(A \to B \to C)$ in a Waldhausen category $\mathscr{C}$. Then there are three functors, $s, t, q : \mathcal{E}(\mathscr{C}) \to \mathscr{C}$ respectively picking out each of the three objects in any exact sequence.

Theorem 4.1. *(Additivity)* If $F' \to F \to F''$ is an exact sequence of functors between Waldhausen categories $\mathscr{C} \to \mathscr{D}$, then $K_n(F) = K_n(F') + K_n(F'')$.

Proof. We remark that giving such an exact sequence of functors is equivalent to giving a functor $\mathscr{C} \to \mathcal{E}(\mathscr{D})$, so we can reduce to proving the statement for the triple $(s,t,q) : \mathcal{E}(\mathscr{D}) \to \mathscr{D}$. We prove that the functor

$$\mathscr{D} \times \mathscr{D} \to \mathcal{E}(\mathscr{D})$$
$$(A,B) \mapsto (A \to A \amalg B \to B)$$

is a homotopy equivalence at the level of K-theory, and the result follows. □

4.2. **Additivity for ∞-categories.** To generalize $\mathcal{E}(\mathscr{C})$ for infinity categories, we want a category whose objects are cofiber sequences. As we can see, this is given by $\mathrm{Gap}_{[2]}(\mathscr{C})$, whose objects we recall are diagrams

$$\begin{array}{ccccc} * & \longrightarrow & X & \longrightarrow & Y \\ & & \downarrow & & \downarrow \\ & & * & \longrightarrow & Z \\ & & & & \downarrow \\ & & & & * \end{array}$$

where $X \to Y \to Z$ is a cofiber sequence. Since a cofiber sequence is determined up to equivalence by the map $f : X \to Y$, we have an equivalence of ∞-categories

$$\mathrm{Fun}(\Delta^1, \mathscr{C}) \simeq \mathrm{Gap}_{[2]}(\mathscr{C}).$$

Let $F : \mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathscr{C} \times \mathscr{C}$ denote the map sending a functor $\Delta^1 \to \mathscr{C}$, whose image is an arrow $X \xrightarrow{\alpha} Y$, to the pair $(X, \mathrm{cofib}(\alpha))$.

Theorem 4.2. *(Additivity)* We have that F induces a homotopy equivalence

$$K(\mathrm{Fun}(\Delta^1, \mathscr{C})) \xrightarrow{\sim} K(\mathscr{C} \times \mathscr{C}) \simeq K(\mathscr{C}) \times K(\mathscr{C}).$$

At the level of quasi-categories, we have that F admits a right homotopy inverse, given by

$$\begin{aligned} \mathscr{C} \times \mathscr{C} &\to \mathrm{Fun}(\Delta^1, \mathscr{C}) \\ (X, Y) &\mapsto (X \to X \amalg Y). \end{aligned}$$

This gives the homotopy inverse at the level of K-spaces.

4.3. **Corollaries of additivity.**

Corollary 4.3. Given a cofiber sequence of functors $F' \to F \to F''$ between pointed ∞-categories admitting finite colimits $\mathscr{C} \to \mathscr{D}$, we have that $K(F) = K(F') + K(F'')$.

Proof. We have three functors $s, t, q : \mathrm{Fun}(\Delta^1, \mathscr{D}) \to \mathscr{D}$ given by taking $X \to Y$ to X, Y, and Y/X, respectively. One can easily see that $K(t) = K(s) + K(q)$.

We see that the natural transformation $F' \to F$ determines a functor $H : \mathscr{C} \to \mathrm{Fun}(\Delta^1, \mathscr{D})$, and we can rewrite $K(F') = K(s) \circ K(H)$, $K(F) = K(t) \circ K(H)$, and $K(F'') = K(q) \circ K(H)$. $\square$

Corollary 4.4. We have that suspension induces a group homomorphism $K(\Sigma) : K_n(\mathscr{C}) \to K_n(\mathscr{C})$ which is multiplication by -1 for every n.

Proof. Apply additivity to the cofiber sequence of morphisms $\mathrm{id} \to * \to \Sigma$. $\square$

5. Universality (Blumberg, Gepner, Tabuada)

5.1. Overview and related results. Results in this section are from [BGT13].

We will attempt to get a handle on what type of enlightening universal property the K-theory of ∞-categories satisfies.

For example given a category $\mathscr{C}$ with some notion of short exact sequences (exact category, triangulated category, Waldhausen category), we can say that an *Euler characteristic* valued in an abelian group A is an assignment of group elements for each isomorphism class in $\mathscr{C}$ which *splits short exact sequences*, that is:

$$0 \to X' \to X \to X'' \to 0 \quad \rightsquigarrow \chi(X) = \chi(X') + \chi(X'').$$

In this sense, $K_0(\mathscr{C})$ is the *universal target group* for Euler characteristics. We would ideally like to extend this universality results to higher K-theory.

5.2. Definitions and notation. Denote by $\mathtt{Cat}_\infty$ the category of small ∞-categories (e.g. quasi-categories).

Denote by $\mathtt{Cat}_\infty^{\mathrm{ex}}$ the (pointed) category of small stable ∞-categories and exact functors (functors which preserve finite limits and colimits).

An ∞-category $\mathscr{C}$ is called *idempotent-complete* if its image under the Yoneda embedding (here the Yoneda embedding is into functors valued in spaces) is closed under retracts. We denote by $\mathtt{Cat}_\infty^{\mathrm{perf}}$ the category of small idempotent-complete stable ∞-categories, so we have an inclusion

$$\mathtt{Cat}_\infty^{\mathrm{perf}} \subseteq \mathtt{Cat}_\infty^{\mathrm{ex}}.$$

This inclusion admits a left adjoint (Higher Topos Theory, 5.1.4.2), which we denote by $\mathrm{Idem} : \mathtt{Cat}_\infty^{\mathrm{ex}} \to \mathtt{Cat}_\infty^{\mathrm{perf}}$.

5.3. Morita equivalence. Two rings R and S are *Morita equivalent* if the categories $\mathtt{Mod}_R$ and $\mathtt{Mod}_S$ are equivalent. This is a *weaker notion* than ring isomorphism, but it is enough to guarantee that the algebraic K-theory of R and S coincide:

$$K(R) \cong K(S).$$

We say two small stable ∞-categories $\mathscr{C}, \mathscr{D} \in \mathtt{Cat}_\infty^{\mathrm{ex}}$ are *Morita equivalent* if $\mathrm{Idem}(\mathscr{C})$ and $\mathrm{Idem}(\mathscr{D})$ are equivalent, and a morphism $\mathscr{C} \to \mathscr{D}$ is a *Morita equivalence* if it induces an equivalence of categories $\mathrm{Idem}(\mathscr{C}) \xrightarrow{\sim} \mathrm{Idem}(\mathscr{D})$.

5.4. Exact sequences. A sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathtt{Cat}_\infty^{\mathrm{perf}}$ (of small stable idempotent-complete infinity categories) is *exact* if:

- the composite is zero
- $\mathcal{A} \to \mathcal{B}$ is fully faithful

- the induced map $\mathcal{B}/\mathcal{A} \to \mathcal{C}$ is an equivalence.

A sequence is *split exact* if it is exact and there exist appropriate adjoint splitting maps.

A sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ in $\mathtt{Cat}_\infty^{\mathrm{ex}}$ (small stable ∞-categories) is (split) exact if the associated sequence

$$\mathrm{Idem}(\mathcal{A}) \to \mathrm{Idem}(\mathcal{B}) \to \mathrm{Idem}(\mathcal{C})$$

is (split) exact in $\mathtt{Cat}_\infty^{\mathrm{perf}}$.

5.5. **Additive and localizing invariants.** Let $E : \mathtt{Cat}_\infty^{\mathrm{ex}} \to \mathscr{D}$ be a functor to a stable presentable* ∞-category $\mathscr{D}$. We say E is an *additive invariant* if it:

- inverts Morita equivalences
- preserves filtered colimits
- sends split exact sequences to cofiber sequences.

We say E is a *localizing invariant* if it sends all exact sequences to cofiber sequences.

Localizing invariants are additive, but the converse does not hold; a counterexample is THH, topological Hochschild homology.

5.6. **Some more notation (sorry).** Let $\mathrm{Fun}_{\mathrm{add}}(\mathtt{Cat}_\infty^{\mathrm{ex}}, \mathscr{D})$ denote the functor category of additive invariants valued in $\mathscr{D}$.

Let $\mathrm{Fun}^L(\mathscr{C}, \mathscr{D})$ be the ∞-category of *colimit-preserving functors*.

Let $\mathcal{S}_\infty$ denote the ∞-category of spectra.

5.7. **The universal additive invariant.** Let $\mathcal{U}_{\mathrm{add}} : \mathtt{Cat}_\infty^{\mathrm{ex}} \to \mathcal{M}_{\mathrm{add}}$ denote the following composite, where $\mathcal{M}_{\mathrm{add}}$ denotes the resulting category:

- apply $\mathrm{Idem} : \mathtt{Cat}_\infty^{\mathrm{ex}} \to \mathtt{Cat}_\infty^{\mathrm{perf}}$
- take the Yoneda embedding y where presheaves are valued in the ∞-category of spectra $\mathcal{S}_\infty$
- restrict to the subcategory of compact objects
- if $\mathscr{A} \to \mathscr{B} \to \mathscr{C}$ is a split exact sequence, localize at maps of the form $y(\mathscr{B})/y(\mathscr{A}) \to y(\mathscr{C})$
- stabilize.

Theorem 5.1. [BGT13, pp. 6.7, 6.10] The functor $\mathcal{U}_{\mathrm{add}}$ is an additive invariant, and moreover is the *universal additive invariant*, in the sense that, for any stable presentable ∞-category $\mathscr{D}$ we have an equivalence of ∞-categories:

$$\mathrm{Fun}^L(\mathscr{M}_{\mathrm{add}}, \mathscr{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathrm{add}}(\mathtt{Cat}_\infty^{\mathrm{ex}}, \mathscr{D}).$$

That is, *every additive invariant* factors through $\mathscr{M}_{\mathrm{add}}$.

5.8. **The application for K-theory.** We should view $\mathscr{M}_{\text{add}}$ as some category of non-commutative motives which is the receptacle for all information about additive invariants. This turns out to be enriched in spectra.

We claim that $\mathcal{S}_\infty^\omega$, the ∞-category of compact spectra, is a stable idempotent-complete ∞-category, that is, it is an element in $\mathtt{Cat}_\infty^{\text{perf}}$. Let $\mathcal{A}$ be any other element of $\mathtt{Cat}_\infty^{\text{perf}}$.

Theorem 5.2. [BGT13, p. 1.3] There is an equivalence of spectra

$$K(\mathscr{A}) \simeq \operatorname{Map}_{\mathscr{M}_{\text{add}}}(\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega), \mathcal{U}_{\text{add}}(\mathscr{A})).$$

In particular for $n \in \mathbb{Z}$ we have an isomorphism of abelian groups

$$K_n(\mathscr{A}) \cong \operatorname{Hom}\left(\mathcal{U}_{\text{add}}(\mathcal{S}_\infty^\omega), \Sigma^{-n}\mathcal{U}_{\text{add}}(\mathcal{A})\right).$$

The suspension functor $\Sigma : \mathscr{M}_{\text{add}} \to \mathscr{M}_{\text{add}}$ turns out to agree with $S_\bullet$ *(BGT, 7.17).*

5.9. **The universal localizing invariant.** An analogous construction may be made to obtain a *universal localizing invariant*

$$\mathcal{U}_{\text{loc}} : \mathtt{Cat}_\infty^{\text{ex}} \to \mathcal{M}_{\text{loc}}.$$

This category $\mathcal{M}_{\text{loc}}$ is analogously some category of non-commutative motives which receives all information about localizing invariants. Its suspension is also given by $S_\bullet$.

Analogous results to those above can be used to describe the *non-connective K-theory* of idempotent-complete stable ∞-categories.

5.10. **Why do we care?** Algebraic K-theory of ∞-categories was *not defined* in terms of universal constructions of presheaves and localizations for infinity categories, so this provides a more universal construction.

The previous result with the Yoneda lemma provides a *total classification of natural transformations from K-theory to other additive (or localizing) invariants.*

This construction provides a tractable formulation of other interesting invariants, for example *topological Hochschild homology*, which is an additive invariant. Via the previous classification we can understand and characterize the trace map $K \to \text{THH}$, an active area of research (see [BGT13, §10]).

Even though topological cyclic homology TC is not an additive or localizing invariant (it doesn't preserve filtered colimits) one can still get a better handle on the cyclotomic trace map $K \to \text{TC}$ (see [BGT13, p. 10.3]).

6. Universality (a la Barwick)

Results in this section are from [Bar16].

6.1. **Slogan.**

> Algebraic K-theory is "a *universal homology theory*, which takes suitable higher categories as input and produces either spaces or spectra as output."

6.2. **Definitions and notation.** For any ∞-category $\mathscr{C}$, we denote by $\iota\mathscr{C}$ its maximal Kan subcomplex.

A *pair of ∞-categories* $(\mathscr{C}, \mathscr{C}_\dagger)$ is an ∞-category $\mathscr{C}$ along with an ∞-subcategory $\mathscr{C}_\dagger$ so that

$$\iota\mathscr{C} \subseteq \mathscr{C}_\dagger \subseteq \mathscr{C}.$$

A morphism of $\mathscr{C}_\dagger$ is called an *ingressive morphism*.

A *functor of pairs* $(\mathscr{C}, \mathscr{C}_\dagger) \to (\mathscr{D}, \mathscr{D}_\dagger)$ is a functor $\mathscr{C} \to \mathscr{D}$ sending ingressive morphisms to ingressive morphisms.

6.3. **Examples of pairs.** For any ∞-category $\mathscr{C}$, there are two trivial pairs:

(1) the *minimal pair*, denoted $\mathscr{C}^\flat$, which is the pair $(\mathscr{C}, \iota\mathscr{C})$, where we recall $\iota\mathscr{C}$ is the maximal Kan subcomplex
(2) the *maximal pair*, denoted $\mathscr{C}^\sharp$, which is the pair $(\mathscr{C}, \mathscr{C})$.

6.4. **Waldhausen ∞-categories.** We say a pair $(\mathscr{C}, \mathscr{C}_\dagger)$ is a *Waldhausen ∞-category* if the following axioms hold:

(1) $\mathscr{C}$ is pointed, and the map $0 \to X$ is ingressive for any X
(2) pushouts of ingressive morphisms exist and are ingressive.

We define a *morphism of Waldhausen ∞-categories* to be any exact functor, by which we mean it:

- preserves zero objects
- sends pushouts along an ingressive morphism to pushouts along an ingressive morphism.

We think (roughly) as the ∞-categorical structure encoding and generalizing weak equivalences, and ingressive morphisms as encoding cofibrations.

We denote by Wald_∞ the ∞-category of Waldhausen ∞-categories (*Barwick, §2*).

6.5. **Examples of Waldhausen ∞-categories.** Equipped with the minimal pair structure $\mathscr{C}^\flat = (\mathscr{C}, \iota\mathscr{C})$, we have a Waldhausen ∞-category if and only if $\mathscr{C}$ is a contractible Kan complex.

With the maximal pair structure $\mathscr{C}^\sharp = (\mathscr{C}, \mathscr{C})$, we have a Waldhausen ∞-category if $\mathscr{C}$ has a zero object and all finite colimits.

Any stable ∞-category equipped with the maximal pair structure is a Waldhausen ∞-category.

If $(\mathscr{C}, \mathscr{C}^{\mathrm{cof}})$ is an *ordinary 1-category with cofibrations*, then its nerve $(N\mathscr{C}, N\mathscr{C}^{\mathrm{cof}})$ is a Waldhausen ∞-category.

6.6. **Theories.** A functor of ∞-categories is *reduced* if it sends the zero object to the terminal object.

Let $\mathcal{E}$ be the category $\mathtt{Kan}$ of Kan complexes (or more generally, any ∞-topos). Then we define a *$\mathcal{E}$-valued theory* to be any reduced functor

$$\phi : \mathrm{Wald}_\infty \to \mathcal{E}$$

which preserves filtered colimits. Denote by

$$\mathrm{Thy}(\mathcal{E}) \subseteq \mathrm{Fun}(\mathrm{Wald}_\infty, \mathcal{E})$$

the full subcategory spanned by $\mathcal{E}$-valued theories.

6.7. **Examples of theories.** The easiest example of a theory $\iota \in \mathrm{Thy}(\mathcal{E})$ is the *interior functor* theory:

$$\begin{aligned} \iota : \mathrm{Wald}_\infty &\to \mathtt{Kan} \\ (\mathscr{C}, \mathscr{C}_\dagger) &\mapsto \iota\mathscr{C}, \end{aligned}$$

sending a Waldhausen ∞-category to its maximal Kan subcomplex.

Give Γ^{op}, the category of finite pointed sets, a set of cofibrations given by monomorphisms of sets with disjoint basepoints. Then $N\Gamma^{\mathrm{op}} \in \mathrm{Wald}_\infty$, and moreover this object *corepresents the interior functor* in the sense that

$$\mathrm{Fun}_{\mathrm{Wald}_\infty}(N\Gamma^{\mathrm{op}}, \mathscr{C}) \xrightarrow{\sim} \iota\mathscr{C}$$

for any $\mathscr{C} \in \mathrm{Wald}_\infty$ [Bar16, Prop. 10.5].

6.8. **Additive theories.** A theory is *additive* if it sends direct sums to products, and a few other technical axioms that are very involved to state [Bar16, pp. 7.4, 7.5]. We think about them as the correct analog, in this setting, of functors splitting exact sequences.

In some sense we would want K-theory to be an additive theory.

Example 6.1. The interior functor $\iota : \mathrm{Wald}_\infty \to \mathtt{Kan}$ is *not additive.*

For theories that fail to be additive, can we provide some additive approximation to them?

Theorem 6.2. *(Additivization)*[Bar16, p. 7.8] Any theory $\phi : \mathrm{Wald}_\infty \to \mathcal{E}$ admits an additivization $D\phi$. Moreover, it is computable as

$$D\phi \simeq \mathrm{colim}_{n\to\infty} \left(\Omega^n_{\mathcal{E}} \circ \Phi \circ \Sigma^n \circ y\right),$$

where y is the map to the derived category $D(\mathrm{Wald}_\infty)$, and Φ is the derived functor of ϕ.

In the sense of Goodwillie calculus, this is the *linearization* of the functor ι.

6.9. **Algebraic K-theory of Waldhausen ∞-categories. Huge definition/theorem:** The *algebraic K-theory functor*

$$K : \mathrm{Wald}_\infty \to \mathtt{Kan}$$

is defined to be the *additivization* of the interior functor $\iota : \mathrm{Wald}_\infty \to \mathtt{Kan}$.

This admits a *canonical delooping*, so we may assume that K-theory is valued in connective spectra (see [Bar16, §7]).

6.10. **Classifying transformations out of K-theory.** Recall that ι was corepresented by $N\Gamma^{\mathrm{op}}$. Combining this fact with the universal property of additivization, we obtain a classification of natural transformations from K-theory to *any other additive theory.*

Proposition 6.3. [Bar16, pp. 10.2, 10.5.1] For any additive theory $\phi : \mathrm{Wald}_\infty \to \mathtt{Kan}$, there is a homotopy equivalence

$$\mathrm{Map}(K, \phi) \simeq \mathrm{Map}(\iota, \phi) = \mathrm{Map}(\mathrm{Fun}_{\mathrm{Wald}_\infty}(N\Gamma^{\mathrm{op}}, -), \phi) \simeq \phi(N\Gamma^{\mathrm{op}}).$$

Corollary 6.4. *([Bar16, p. 10.5.2], Barratt-Priddy-Quillen)* Applying this to $\phi = K$, we get that the *endomorphisms of algebraic K-theory* are

$$\mathrm{End}(K) = K(\Gamma^{\mathrm{op}}) = QS^0 = \mathrm{colim}_n \Omega^n S^n.$$

6.11. **Relation to A-theory.** For any ∞-topos $\mathcal{E}$, we can take its ∞-category of pointed compact objects $\mathcal{E}_*^\omega$. Its algebraic K-theory

$$K(\mathcal{E}_*^\omega)$$

is called the *A-theory of $\mathcal{E}$*.

For $X \in \mathtt{Kan}$, we have an ∞-topos $\mathrm{Fun}(X, \mathtt{Kan})$, and we have that

$$K(\mathrm{Fun}(X, \mathtt{Kan})) = A(X)$$

agrees with the A-theory of X that we have seen.

7. Conclusion

7.1. **Conclusion.** Infinity categories are the most general setting for a study of algebraic K-theory.

Universal constructions of algebraic K-theory provide a framework for the analysis of K-theory and other theories like A-theory.

Representability results allow a more tangible grasp of interactions between algebraic K-theory and other related theories like THH and TC, as well as trace maps between these.

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