

\mathbb{A}^1 -Homotopy Theory from an $(\infty,1)$ -Categorical Viewpoint

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Abstract

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1 Introduction

1.1 Assumptions

We take Sm_S to be the category of smooth schemes of finite type over a base scheme S , where S is always assumed to be Noetherian and finite dimensional, unless otherwise indicated.

Throughout these notes, “ ∞ -category” refers to a quasi-category.

1.2 Motivation

We want to do homotopy theory in Sm_S [KV71]. If we have two maps $f : X \rightarrow Y$ and $g : X \rightarrow Y$, we might say that they are \mathbb{A}^1 -homotopic if they factor as

$$\begin{array}{ccc} X & & \\ \downarrow & \searrow & \\ X \times \mathbb{A}^1 & \xrightarrow{\quad} & Y \\ \uparrow & \nearrow & \\ X & & \end{array}$$

however this is a little too naive. For example, it is not transitive on hom-sets.

Therefore one considers “universal \mathbb{A}^1 -homotopy theory on Sm_S ,” and then one imposes the wanted relations by “localization.” Consider the following categories

$$\mathrm{Sm}_S \xrightarrow{\text{Yoneda}} \mathrm{PSh}(\mathrm{Sm}_S) \leftrightarrow \mathrm{Sh}(\mathrm{Sm}_S)$$

we have two full subcategories $\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_S), \mathrm{PSh}^{\mathbb{A}^1}(\mathrm{Sm}_S)$ of $\mathrm{PSh}(\mathrm{Sm}_S)$, whose inclusions have left adjoints $L_{\mathrm{Nis}} : \mathrm{PSh}(\mathrm{Sm}_S) \rightarrow \mathrm{Sh}(\mathrm{Sm}_S)$, and $L_{\mathbb{A}^1} : \mathrm{PSh}(\mathrm{Sm}_S) \rightarrow \mathrm{PSh}^{\mathbb{A}^1}(\mathrm{Sm}_S)$. Taking the intersection, we get $\mathrm{Sh}^{\mathbb{A}^1}(\mathrm{Sm}_S) = \mathrm{Sh}(\mathrm{Sm}_S) \cap \mathrm{PSh}^{\mathbb{A}^1}(\mathrm{Sm}_S)$, which is the unstable \mathbb{A}^1 -homotopy category.

We additionally obtain a map $\underline{\quad} : \mathrm{Spc} \rightarrow \mathrm{PSh}(\mathrm{Sm}_S)$ which interprets a space as a constant presheaf over our category Sm_S . This will be defined further later.

An example of a calculation in our category would be the pushout

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\circ} & \mathbb{A}^1 \simeq * \\ \downarrow \circ & & \downarrow \\ \mathbb{A}^1 \simeq * & \longrightarrow & \mathbb{P}^1 \end{array}$$

which shows that $\underline{S}^1 \wedge \mathbb{G}_m \simeq \mathbb{P}^1$, this is effectively (up to \mathbb{A}^1 -contractibility) a loop object in our category.

1.3 Why the Nisnevich topology?

The big reason is that algebraic K -theory does not satisfy étale descent, so the étale topology is not right for our theory.

We could try the Zariski topology, but this ends up being too coarse. For manifolds X , we can form the homotopy quotient

$$X/X \setminus \{x\} = \mathbb{R}^n / \mathbb{R}^n \setminus \{0\} \simeq S^n.$$

This doesn't work in the Zariski topology.

Example 1.3.1. Take a field $k = \bar{k}$, $S = \text{Spec}(k)$, and $x \in X \in \text{Sm}_S$ a closed point.

$$\begin{array}{ccc} X & \xleftarrow{o} & U \\ \downarrow & & \downarrow^{et} \\ S & \xleftarrow{\quad} & \mathbb{A}_S^m \end{array}$$

so subtracting a point we get

$$\begin{array}{ccccc} U \setminus \{x\} & \xleftarrow{o} & U & \dashrightarrow & U/U \setminus \{x\} \\ \downarrow^o & & \downarrow^o & & \downarrow^{\simeq} \\ X \setminus \{x\} & \xleftarrow{o} & X & \dashrightarrow & X/X \setminus \{x\} \end{array}$$

Shrink U such that only x is in the preimage of $0 \in \mathbb{A}_S^m$. Then we have that

$$\begin{array}{ccccc} U \setminus \{*\} & \xleftarrow{o} & U & \dashrightarrow & U/U \setminus \{x\} \\ \downarrow & & \downarrow & & \downarrow^{\simeq} \\ \mathbb{A}_S^m \setminus \{0\} & \xrightarrow{o} & \mathbb{A}_S^m & \dashrightarrow & \mathbb{A}_S^m / \mathbb{A}_S^m \setminus \{0\} \end{array}$$

is a Nisnevich square.

2 Preliminaries

A good definition for our ∞ -category of spaces is $\text{Spc} = N_\Delta(\text{Kan})$.

We have various constructions of new ∞ -categories from old ones:

- full subcategories spanned by objects
- functor categories.

There is a fully faithful functor from a small infinity category \mathcal{C} as follows

$$\begin{aligned} \mathcal{C} &\xrightarrow{y} \text{Fun}(\mathcal{C}^{\text{op}}, \text{Spc}) = \text{PSh}(\mathcal{C}) \\ X &\mapsto \text{Map}_{\mathcal{C}}(-, X) \end{aligned}$$

where y is called the *Yoneda embedding* for \mathcal{C} an (essentially) small ∞ -category [Lur09, 5.1.3.1].

We also get that

$$\text{Map}_{\text{PSh}(\mathcal{C})}(y(X), E) \simeq E(X).$$

Every $E \in \text{PSh}(\mathcal{C})$ is a colimit of representables

$$E \simeq \text{colim}_{y(U) \rightarrow E} y(U)$$

and y preserves all small colimits in \mathcal{C} . Note that limits and colimits are given “objectwise” [Lur09, 5.1.2.3].

An ∞ -category \mathcal{C} is called *presentable* if it is cocomplete and *accessible*, where accessible means that

$$\mathrm{Ind}_\kappa(\mathcal{C}') \xrightarrow{\simeq} \mathcal{C}$$

for a small ∞ -category \mathcal{C}' . We could think of this as meaning that there is a small subcategory $\mathcal{C}^0 \subseteq \mathcal{C}$, which generates \mathcal{C} under sufficiently filtered colimits. We refer to [Lur09, §5.4] for more information.

Example 2.0.1.

- Spc is presentable [Lur09, 5.3.5.12].
- $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is presentable if \mathcal{D} is presentable (and \mathcal{C} is small). In particular, for any small ∞ -category \mathcal{C} , the presheaf category $PSh(\mathcal{C})$ is presentable. In fact, every presentable category arises as the localization of a presheaf category on a small ∞ -category [Lur09, 5.5.1.1].

Presentable ∞ -categories have all limits [Lur09, 5.5.2.4].

Theorem 2.0.2. (Adjoint functor theorem [Lur09, 5.5.2.9]) If $F : \mathcal{C} \rightarrow \mathcal{D}$ is a functor between presentable infinity categories, then

1. F has a right adjoint if and only if F preserves all small colimits.
2. F has a left adjoint if and only if F preserves all small limits and is accessible.

Definition 2.0.3. Let \mathcal{C} be a presentable ∞ -category, and M a set of morphisms in \mathcal{C} . An object $Z \in \mathcal{C}$ is called *M-local* if the associated map

$$\mathrm{Map}(Y, Z) \xrightarrow{f^*} \mathrm{Map}(X, Z)$$

is an equivalence for all $f \in M$. The full subcategory \mathcal{C}' spanned by all M -local objects is representable and fits into an adjunction

$$L : \mathcal{C} \rightleftarrows \mathcal{C}' : \mathrm{incl}$$

where $\mathrm{incl} : \mathcal{C}' \subseteq \mathcal{C}$ is the inclusion map [Lur09, 5.5.4.15].

2.1 Sheaves

A large issue is that $y : \mathrm{Sm}_S \hookrightarrow PSh(\mathrm{Sm}_S)$ does not preserve colimits. To see this, take $\coprod_\alpha U_\alpha \in \mathrm{Sm}_S$, and $A \in Spc$, then

$$\mathrm{Map}\left(\coprod_\alpha y(U_\alpha), \underline{A}\right) \simeq \prod \mathrm{Map}(y(U_\alpha), \underline{A}) \simeq \prod \underline{A}(U_\alpha) \simeq \prod A$$

by the Yoneda lemma and the fact that \underline{A} is a constant sheaf. However if we take the coproduct inside, we see

$$\mathrm{Map}\left(y\left(\coprod_\alpha U_\alpha\right), \underline{A}\right) \simeq \underline{A}\left(\coprod_\alpha U_\alpha\right) \simeq A.$$

Definition 2.1.1. Let \mathcal{C} always be an (essentially) small category. For a morphism $Y \xrightarrow{f} X$ in an infinity category \mathcal{P} with fiber products, the *Čech nerve* is a simplicial object

$$\check{C}(f)_\bullet \in \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{P})$$

whose evaluation at $[n] \in \Delta^{\text{op}}$ is given by the $(n + 1)$ -fold fiber product

$$\check{C}(f)_n = Y \times_X \cdots \times_X Y.$$

This comes with an augmentation

$$\check{C}(f)_\bullet \rightarrow \underline{X}$$

which becomes, after taking the geometric realization (a colimit),

$$|\check{C}(f)_\bullet| \rightarrow X.$$

Definition 2.1.2. Suppose we are given for every object $X \in \mathcal{C}$ a collection $\mathcal{T}'(X) = \{U_\alpha \xrightarrow{f_\alpha} X\}_\alpha$ of \mathcal{C} -morphisms. Then $E \in PSh(\mathcal{C})$ is called a (*Čech*) *sheaf* for \mathcal{T}' if for every $X \in \mathcal{C}$ and every $f_\alpha \in \mathcal{T}'(X)$:

$$\text{Map}(X, E) \rightarrow \text{Map}(|\check{C}(f_\alpha)_\bullet|, E)$$

is an equivalence. Equivalently, E is a Čech-sheaf if it is M -local with respect to

$$M := \{|\check{C}(f_\alpha)_\bullet| \rightarrow X : X \in \mathcal{C}, f_\alpha \in \mathcal{T}'(X)\},$$

in which case the Bousfield localization yields an adjunction

$$L : PSh(\mathcal{C}) \rightleftarrows Sh(\mathcal{C}) : i.$$

Here we call L the *Čech-sheafification* for \mathcal{T}' .

Definition 2.1.3. A family $\mathcal{U} = \{U_\alpha \xrightarrow{p_\alpha} X\}$ in Sm_S is called a *Nisnevich covering* on X if

- Every p_α is étale
- For every field k and $\text{Spec}(k) \rightarrow X$, there exists a lift

$$\begin{array}{ccc} & & U_\alpha \\ & \nearrow \text{dashed} & \downarrow p_\alpha \\ \text{Spec}(k) & \longrightarrow & X \end{array}$$

Or, trivially, we obtain a Nisnevich covering if $X = \emptyset$ or if $\mathcal{U} = \emptyset$.

We now have $PSh(\text{Sm}_S) \rightleftarrows Sh(\text{Sm}_S)$. We could further localize at morphisms of the form $y(U \times \mathbb{A}^1) \rightarrow y(U)$, and obtain the category $Sh^{\mathbb{A}^1}(\text{Sm}_S)$ of \mathbb{A}^1 -invariant sheaves.

Definition 2.1.4. A *sieve* R on $X \in \mathcal{C}$ is a full subcategory of \mathcal{C}/X such that if

$$\begin{array}{ccc} Z & \longrightarrow & Y \\ & \searrow & \downarrow \\ & & X \end{array}$$

is a morphism in \mathcal{C}/X and $(Y \rightarrow X) \in R$, then $(Z \rightarrow X) \in R$.

Proposition 2.1.5. [Lur09, 6.2.2.5, 6.2.3.4, 6.2.3.18]

Let $\mathcal{U} = \{U_\alpha \rightarrow X\} \in \mathcal{T}^1(X)$, and let

$$R_{\mathcal{U}} := \{Y \xrightarrow{f} X : f \text{ factors through some } U_\alpha \rightarrow X\}.$$

Then there is a bijection

$$\begin{aligned} \{\text{Sieves on } X\} &\leftrightarrow \{\text{subobjects } [U \rightarrow y(X)] \in \text{PSh}(\mathcal{C})\} \\ \{Y \rightarrow X : \exists! y(U) \xrightarrow{!} U \rightarrow y(X)\} &\leftrightarrow [U \rightarrow y(X)] \\ R &\mapsto [\text{colim}_{(Y \rightarrow X) \in R} y(U) \rightarrow y(X)] \\ R_{\mathcal{U}} &\mapsto [|\check{\mathcal{C}}(U)| \rightarrow y(X)]. \end{aligned}$$

Let τ be a (Grothendieck) topology on \mathcal{C} . Let (\mathcal{C}, τ) be a site. Then a presheaf $E \in \text{PSh}(\mathcal{C})$ is called a *sheaf* (for τ) if for every $X \in \mathcal{C}$ and every $R \in \tau(X)$ with a correlating subobject $[U \hookrightarrow y(X)]$, the map

$$\text{Map}(X, E) \rightarrow \text{Map}(U, E)$$

is an equivalence. Let τ' be a collection as above. We have an axiom from [Hoy17, C1] given as

(PT2) If $f : X' \rightarrow X$ is a morphism and $\{U_\alpha \rightarrow X\} \in \tau'(X)$, then $\{f^* : U_\alpha \rightarrow X'\} \in \tau'(X')$

Let τ denote the generated topology. Then for $E \in \text{PSh}(\mathcal{C})$, we have that E is a sheaf for τ if and only if E is a (Čech) sheaf for τ' .

Remark. The Nisnevich coverings satisfy (PT2) 2.1.

Definition 2.1.6. A *Nisnevich distinguished square* is a pullback square Q in Sm_S of the form

$$\begin{array}{ccc} V & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow p_Q \\ U & \xrightarrow{j_Q} & X \end{array}$$

where we have that

1. j_Q is an open immersion
2. p_Q is étale
3. p_Q induces an isomorphism $Y \setminus V \xrightarrow{\cong} X \setminus U$.

Definition 2.1.7. The collection P of those squares is called the *Nisnevich-cd-structure*, where the associated topology τ_P is the topology on Sm_S generated by

$$R_{\{j_Q, p_Q\}} \in \tau(X) \quad R_\emptyset \in \tau(\emptyset).$$

Theorem 2.1.8. (Voevodsky, [Hoy17, Prop I, 3.2.5]) A presheaf $E \in \text{PSh}(\text{Sm}_S)$ is a τ_P -*sheaf* if and only if E is *P-excisive*. That is,

1. E takes each distinguished square $Q \in P$ to a pullback square $E(Q)$ in Spc .
2. $E(\emptyset) = *$.

Remark.

- $\tau_P \subseteq \tau$.
- Let k be a field of characteristic $\neq 2$, and let $a \neq 0$ in k , then consider the two maps

$$\begin{aligned} \mathbb{A}_k^1 \setminus \{a\} &\xrightarrow{\circ} \mathbb{A}_k^1 \\ \mathbb{A}_k^1 \setminus \{0\} &\rightarrow \mathbb{A}_k^1 \\ z &\mapsto z^2. \end{aligned}$$

This is a Nisnevich covering if a is a square in the field k . And if we remove one of the two roots, say b , of a , we obtain a Nisnevich square

$$\begin{array}{ccc} \mathbb{A}^1 \setminus \{*\} & \dashrightarrow & \mathbb{A}_k^1 \setminus \{0, b\} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{A}^1 \setminus \{a\} & \longrightarrow & \mathbb{A}^1. \end{array}$$

Definition 2.1.9. A map $E \rightarrow F$ in $PSh(\text{Sm}_S)$ is a *local equivalence* if $L(E \rightarrow F)$ is an equivalence.

Definition 2.1.10. A *Nisnevich neighborhood* of a point $x \in X \in \text{Sm}_S$ is a pair $(Y \xrightarrow{f} X, y)$ consisting of an étale map p where Y is connected and a lift

$$\begin{array}{ccc} & & Y \\ & \nearrow y & \downarrow \\ \text{Spec}(k(x)) & \longrightarrow & X \end{array}$$

Morphisms of those are morphisms over X respecting the lifts. This defines a category $\mathcal{I}_{(X,x)}$.

Lemma 2.1.11. $\mathcal{I}_{(X,x)}^{\text{op}}$ is filtered. The collection $p = (X, x) : \mathcal{I}_{(X,x)}^{\text{op}} \rightarrow \text{Sm}_S$ via $(Y, y) \mapsto y$ of filtered diagrams is called the *standard Nisnevich points* (NPts). And the *stalk* functor at $p \in \text{NPts}$ is

$$\begin{aligned} (-)_P &: PSh(\text{Sm}_S) \rightarrow \text{Spc} \\ E &\mapsto \text{colim}_{(Y,c) \in \mathcal{I}_{(X,x)}^{\text{op}}} E(y) \end{aligned}$$

A map $E \rightarrow F$ in $PSh(\text{Sm}_S)$ is called a *stalkwise equivalence* if $E_p \rightarrow F_p$ is an equivalence for all $p \in \text{NPts}$.

Example 2.1.12. For $E \in Sh(\text{Sm}_S)$ we have that

$$E \simeq \emptyset \Leftrightarrow \forall p \in \text{NPts}, E_p = \emptyset$$

Remark. The limit $\lim_{(Y,y) \in \mathcal{I}_{(X,x)}} Y$ exists as a scheme and is given by the Henselization $\mathcal{O}_{X,x}^h$ of the local ring $\mathcal{O}_{X,x}$.

Warning. The “quasi-representable” presheaf of spaces

$$\tilde{y}(\text{Spec}(\mathcal{O}_{X,x}^h)) := \text{Map}_{\text{Sch}/S}(i(-), \text{Spec}(\mathcal{O}_{X,x}^h)) \in PSh(\text{Sm}_S)$$

where $i : \text{Sm}_S \hookrightarrow \text{Sch}/S$, has the issue that usually, the stalks do not agree, i.e. $E_{(X,x)} \not\cong \text{Map}(\tilde{y}(\text{Spec}(\mathcal{O}_{X,x}^h)), E)$.

2.2 Local equivalences are equivalent to stalkwise equivalences

We would like to prove that local equivalences of presheaves are equivalent to stalkwise equivalences. In order to do this, we first must discuss truncated objects.

Definition 2.2.1. Let $n \geq -1$ an integer, and Σ an ∞ -category. We say that $X \in \mathit{Spc}$ is *n-truncated* if $\pi_i(X, x) = *$ for all $i > n$ and for each basepoint $x \in X$.

Extending this definition, we say that an element of an ∞ -category $X \in \Sigma$ is *n-truncated* if $\mathrm{Map}(A, X)$ is *n-truncated* for all $A \in \Sigma$.

Let $\tau_n \Sigma$ be the full subcategory on these objects. If Σ is presentable, then there exists a left adjoint

$$L_m : \Sigma \rightleftarrows \tau_n \Sigma : \mathrm{incl}$$

[Lur09, 5.5.6.18]

A limit (in Spc) of *n-truncated* spaces is again *n-truncated* [Lur09, 5.5.6.5], therefore $E \in \tau_n \mathit{Sh}(\mathrm{Sm}_S)$ if and only if $E \in \mathit{Sh}_{\tau_n}(\mathrm{Sm}_S)$ (i.e. E is objectwise *n-truncated* and a sheaf).

If $n \geq 0$, then $\pi_n : \tau_n \mathit{Spc}_* \rightarrow \mathit{Set}_*$ preserves limits.

Theorem 2.2.2. (Voevodsky) Recall that S is a Noetherian, finite dimensional scheme. For a map of presheaves $f : E \rightarrow F$ in $\mathit{PSh}(\mathrm{Sm}_S)$, we have that f is a local equivalence if and only if it is a stalkwise equivalence.

Proof.

\Rightarrow : complicated

\Leftarrow : For $n \geq 0$, $E \in \mathit{PSh}$, $X \in \mathrm{Sm}_S$, and $x \in E(X)$ a section, we consider the presheaf

$$\begin{aligned} \pi_n^X(E, x) : (\mathrm{Sm}_S/X)^{\mathrm{op}} &\rightarrow \mathit{Set} \\ U &\mapsto \pi_n(E(U), X|_n). \end{aligned}$$

For $E \in \tau_n \mathit{Sh}$, this is a sheaf.

Hence $E \in \tau_n \mathit{Sh}$ and the fact that all $\pi_m^X(E, x) \simeq *$ implies that $E \in \tau_{n-1} \mathit{Sh}$. So inductively, a stalkwise equivalence in $\tau_n \mathit{Sh}$ is a local equivalence.

Definition 2.2.3. For $E \in \Sigma = \mathit{Sh}$, we have the tower

$$\begin{array}{c} \begin{array}{c} E \\ \downarrow \\ \lim_{n \in \mathbb{N}^{\mathrm{op}}} L_n(E) \end{array} \xleftarrow{t_E} \begin{array}{c} \cdots \\ \longrightarrow L_n(E) \end{array} \longrightarrow L_{n-1}(E) \longrightarrow \cdots \longrightarrow L_{-1}(E) \end{array}$$

called the *Postnikov tower of E*. We say that these *converge in Σ* if t_E is an equivalence for all $E \in \Sigma$.

In particular, we have that this is true for $\mathit{Sh}(\mathrm{Sm}_S)$ [Lur04, XI - Descent theorems]. Therefore a stalkwise equivalence is a local equivalence. We show this as follows:

$$\begin{array}{ccccccc}
& & & & & & F \\
& & & & & & \downarrow \\
& & & & & & L_n(F) \longrightarrow L_{n-1}(F) \longrightarrow \cdots \\
& & & & & & \uparrow \\
& & & & & & E \\
& & & & & & \downarrow \\
& & & & & & L_n(E) \longrightarrow L_{n-1}(E) \longrightarrow \cdots \\
& & & & & & \uparrow \\
& & & & & & L_n(F) \\
& & & & & & \uparrow \\
& & & & & & \lim L_n(F) \\
& & & & & & \uparrow \\
& & & & & & t_E \\
& & & & & & \lim_{n \in \mathbb{N}^{\text{op}}} L_n(E) \longrightarrow \cdots \longrightarrow L_n(E) \longrightarrow L_{n-1}(E) \longrightarrow \cdots \longrightarrow L_{-1}(E)
\end{array}$$

Let $E \rightarrow F$ be a stalkwise equivalence, then the objectwise truncation functor gives that $L_n^{\text{obj}}(E) \rightarrow L_n^{\text{obj}}(F)$ is again a stalkwise equivalence. Now $L_n = LL_n^{\text{obj}}$ (via the explicit formula for E). Then we see that $G \rightarrow LG$ is a stalkwise equivalence, so $L_n(E) \rightarrow L_n(F)$ is a stalkwise equivalence. This is a local equivalence by the argument before. Note that a limit of equivalences is then an equivalence, and therefore by equivalence, this is a local equivalence $E \rightarrow F$ that we started with. \square

2.3 Adjoint functors on presheaves

Remark. (Gluing for étale sheaves) Let S be a scheme, with $U \xrightarrow{j} S \xleftarrow{i} Z = S \setminus U$ where j is an open immersion. For any abelian sheaf F on $S_{\text{ét}}$, there is an exact sequence

$$0 \rightarrow j_!j^*F \rightarrow F \rightarrow i_*i^*F \rightarrow 0,$$

which may be verified by checking on stalks. We can also reformulate this in the following way,

$$\begin{array}{ccc}
j_!j^*F & \longrightarrow & F \\
\downarrow & \lrcorner & \downarrow \\
0 & \longrightarrow & i_*i^*F
\end{array}$$

is both a pushout and a pullback square.

Ideally, we would like an analog of this functor for $F \in Sh(\text{Sm}_S)$.

Basic functoriality: let $f : T \rightarrow S$ be a morphism of base schemes (which are Noetherian and finite-dimensional). We get a functor $\text{Sm}_S \rightarrow \text{Sm}_T$, where $X \mapsto X \times_S T =: X_T$. This induces

$$f_* : PSh(\text{Sm}_T) \rightarrow PSh(\text{Sm}_S) \quad (f_*F)(X) = F(X_T)$$

Note. f_* preserves all limits and colimits (in particular filtered colimits, so it is accessible). By the Adjoint Functor Theorem 2.0.2, this implies that f_* has a left adjoint $PSh(\text{Sm}_S) \rightarrow PSh(\text{Sm}_T)$, which we denote by f_{pre}^* . (Note that this does not preserve sheaves in general).

If $\mathcal{U} = \{U_\alpha \rightarrow X\}$ is a Nisnevich covering of $X \in \text{Sm}_S$, then $\mathcal{U} \times_S T := \{U_\alpha \times_S T \rightarrow X_T\}$ is also a Nisnevich covering. This implies that f_* preserves sheaves.

Thus we have a functor $f_* : Sh(\text{Sm}_T) \rightarrow Sh(\text{Sm}_S)$.

Lemma 2.3.1. $f_* : Sh(\text{Sm}_T) \rightarrow Sh(\text{Sm}_S)$ also has a left adjoint, given by the composite

$$Sh(\text{Sm}_S) \xleftarrow{\text{incl}} PSh(\text{Sm}_S) \xrightarrow{f_{pre}^*} PSh(\text{Sm}_T) \xrightarrow{L} Sh(\text{Sm}_T).$$

Proof. For $G \in Sh(\text{Sm}_T)$, we have

$$\begin{aligned} \text{Map}_{Sh}(f^*F, G) &= \text{Map}_{Sh}(L \circ f_{pre}^* \circ \text{incl}(F), G) \simeq \text{Map}_{PSh}(f_{pre}^* \circ \text{incl}(F), \text{incl}(G)) \\ &\simeq \text{Map}_{PSh}(\text{incl}(F), f_* \circ \text{incl}(G)) \simeq \text{Map}_{Sh}(F, f_*(G)) \end{aligned}$$

where the last line holds because $PSh \leftrightarrow Sh$ is a full subcategory. \square

Lemma 2.3.2. For $X \in \text{Sm}_S$, we have that $f_{pre}^*(X) \simeq X \times_S T \simeq f^*(X)$

Proof. For any $F \in PSh(\text{Sm}_T)$, we check that

$$\begin{aligned} \text{Map}(f^*X, F) &\stackrel{adj}{\simeq} \text{Map}(X, f_*F) \stackrel{Yon.}{\simeq} (f_*F)(X) \stackrel{defn}{\simeq} F(X \times_S T) \\ &\simeq \text{Map}(X \times_S T, F) \end{aligned}$$

\square

Now assume, in addition, that $f : T \rightarrow S$ is smooth. We then obtain the following functor:

$$\begin{aligned} \text{Sm}_T &\rightarrow \text{Sm}_S \\ (X/T) &= (X \rightarrow T) \mapsto (X \rightarrow T \xrightarrow{f} S) = (X/S). \end{aligned}$$

This induces

$$f^\# : PSh(\text{Sm}_S) \rightarrow PSh(\text{Sm}_T) \quad (f^\#F)(X/T) = F(X/S).$$

Note. $f^\#$ has a left adjoint $f_{\#}^{\text{pre}}$. Additionally, $f^\#$ preserves sheaves, so we obtain an adjunction at the level of sheaves

$$f_{\#} : Sh(\text{Sm}_T) \rightleftarrows Sh(\text{Sm}_S) : f^\#.$$

Where we define $f_{\#} := L \circ f_{\#}^{\text{pre}} \circ \text{incl}$.

Exercise 2.3.3. For $X \in \text{Sm}_T$, we have $f_{\#}^{\text{pre}}(X/T) \simeq f_{\#}(X/T) \simeq X/S$.

Lemma 2.3.4. If f is smooth, then

$$f^\# \simeq f_{pre}^* : PSh(\text{Sm}_S) \rightarrow PSh(\text{Sm}_T).$$

In particular, f_{pre}^* preserves sheaves and $f^* \simeq f_{pre}^*|_{Sh(\text{Sm}_S)}$.

Proof. Since $f^\#$ and f^* both commute with colimits, and since each presheaf is built as a colimit of representables, it suffices to show this on representable presheaves, that is, $f^\#(X/S) \simeq f_{pre}^*(X/S)$. Note that we have already computed $f_{pre}^*(X/S) \simeq X \times_S T$. So for any $U \in \text{Sm}_T$, we have that

$$f^\#(X/S)(U/T) \stackrel{Yon.}{\simeq} \text{Map}((U/T), f^\#(X/S)) \stackrel{adj}{\simeq} \text{Map}(f_{pre}^*(U/T), X/S) \simeq \text{Map}(U/T, X \times_S T).$$

So for f smooth, we will identify $f^* \simeq f^\#$. \square

2.4 Towards the gluing theorem

As before, let $U \xrightarrow{j} S \xleftarrow{i} Z = S \setminus U$ where j is an open immersion, and i is closed. We recall the induced maps

$$\begin{array}{ccc} PSh(\mathrm{Sm}_S) & \xrightleftharpoons{j_*, j_\#} & PSh(\mathrm{Sm}_U) \\ i^*, i_\# \downarrow & \uparrow i_*, i_\# & \uparrow j^*, j_\# \\ PSh(\mathrm{Sm}_Z) & & \end{array}$$

Take $F \in Sh(\mathrm{Sm}_S)$. Then we have a square of the form

$$\begin{array}{ccc} j_\# j^* F & \xrightarrow{\text{counit}} & F \\ \downarrow & & \downarrow \text{unit} \\ U = j_\#(U/U) & \longrightarrow & i_* i^* F \end{array}$$

where the map along the left is $j_\#(j^* F \rightarrow U \simeq *)$, as we note that U is the final object in $Sh(\mathrm{Sm}_U)$. There is a unique map $\emptyset \xrightarrow{!} i^* F$, so by adjunction we get the bottom map $U \rightarrow i_* i^* F$.

Q: Is the above square (labeled G) a pushout square?

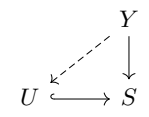
Assume that the sheaf F is represented by some $X \in \mathrm{Sm}_S$. Then we have $j^* X \simeq X \times_S U \simeq X_U$, $j_\#(X_U) \simeq (X_U/S)$, and $i^* X = X \times_S Z = X_Z$, so we are asking whether the following square is a pushout

$$\begin{array}{ccc} X_U & \longrightarrow & X \\ \downarrow & & \downarrow \\ U & \longrightarrow & i_*(X_Z) \end{array}$$

that is, whether $U \coprod_{X_U} X \rightarrow i_*(X_Z)$ is an equivalence. So we first look at it in presheaves, then check if it is an equivalence after sheaffication (a local equivalence).

We first compute the left hand side. For any $Y \in \mathrm{Sm}_S$, we have

$$(U \coprod_{X_U}^{\mathrm{pre}} X)(Y) \simeq U(Y) \coprod_{X_U(Y)} X(Y) \simeq \begin{cases} X(Y) & U(Y) = \emptyset \\ * & U(Y) = *, \text{ that is,} \end{cases}$$



which implies the square factors through $Y \rightarrow X_U$

and the right hand side is computed as

$$(i_* X_Z)(Y) \simeq X_Z(Y_Z) \simeq X(Y_Z) = * \quad \text{if } U(Y) = *$$

however we must see what happens if $U(Y) = \emptyset$. Assume S is irreducible, $U \notin \{\emptyset, S\}$, and look at the stalks at a point $z \in Z \subset S$:

$$\begin{array}{ccc}
(U \amalg_{X_U} X)_{S,z} & \xlongequal{\quad} & X(\mathrm{Spec}(\mathcal{O}_{S,z}^h)) \\
& & \downarrow (*) \\
(i_* X_Z)_{S,z} & \xlongequal{\quad} & X(\mathrm{Spec}(\mathcal{O}_{S,z}^h) \times_S Z)
\end{array}$$

This map $(*)$ is an isomorphism if $X \rightarrow S$ is étale, but in general it is not an isomorphism, e.g. if $X = \mathbb{A}_S^1$.

Conclusion: the square (G) is generally not a pushout.

In the square:

$$\begin{array}{ccc}
j_{\#} j^* & \xrightarrow{\text{counit}} & F \\
\downarrow & & \downarrow \text{unit} \\
U & \longrightarrow & i_* i^* F
\end{array}$$

all the functors $(j_{\#}, j^*, i^*)$ are left adjoints and commute with colimits, except for i_* . We will see that it somehow commutes with “enough” colimits.

Lemma 2.4.1. Let $i : Z \hookrightarrow S$ be a closed immersion. Then for $F \in PSh(\mathrm{Sm}_Z)$, such that $F(\emptyset) = *$, we get a map $Li_*(F) \rightarrow i_* L(F)$ is an equivalence.

Proof. We can check on stalks again. Take any $X \in \mathrm{Sm}_S$, and $x \in X$. We then have two cases

Case 1: $x \notin X \times_S Z \xrightarrow{X}$

Case 2: $x \in X_Z \xrightarrow{X}$.

In Case 1, the Nisnevich neighborhoods (V, v) of (X, x) with the property that $V \times_S Z = \emptyset$ are cofinal in all Nisnevich neighborhoods. So we compute the stalks

$$\begin{aligned}
Li_*(F)_{X,x} &\simeq i_*(F)_{X,x} \simeq \operatorname{colim}_{\substack{(V,v) \\ \text{as above}}} i_*(F)(V) \\
&\simeq \operatorname{colim}_{(V,v)} F(V \times_S Z) \simeq \operatorname{colim}_{(V,v)} F(\emptyset) \simeq \operatorname{colim}_{(V,v)} * \simeq *.
\end{aligned}$$

We can also go backwards to see that

$$* \simeq \dots \simeq (i_* L(F))_{X,x}.$$

In Case 2, this is an exercise. Use that any Nisnevich neighborhood of $x \in X_Z$ can be extended to a Nisnevich neighborhood of $x \in X$. Then use cofinality argument as in the case above. \square

Proposition 2.4.2. For $i : Z \hookrightarrow S$ closed immersion, then $i_* : Sh(\mathrm{Sm}_Z) \rightarrow Sh(\mathrm{Sm}_S)$ commutes with *weakly contractible colimits*. (The geometric realization of the indexing category is weakly equivalent to a point, e.g. filtered colimits)

Proof. A colimit in Sh is computed as $L \circ \text{colim}^{\text{pre}}$. Note the geometric realization of A is $\text{colim}_{\alpha \in A}$, so weak contractibility gives that this is a point, and thus that we can reverse L and i_* .

$$\begin{aligned} i_* \circ \text{colim}_{\alpha \in A} F_\alpha &\simeq i_* \circ L \circ \text{colim}_{\alpha \in A}^{\text{pre}} F_\alpha \stackrel{\text{Lem.}}{\simeq} L \circ i_* \circ \text{colim}_{\alpha \in A}^{\text{pre}} F_\alpha \\ &\simeq L \circ \text{colim}_{\alpha \in A}^{\text{pre}} i_*(F_\alpha) \simeq \text{colim}_{\alpha \in A} i_*(F_\alpha). \end{aligned}$$

□

Now back to smooth morphisms $f : T \rightarrow S$.

Consider a pullback square in schemes of the following form

$$\begin{array}{ccc} \tilde{T} & \xrightarrow{\tilde{f}} & \tilde{S} \\ \tilde{p} \downarrow & & \downarrow p \\ T & \xrightarrow{f} & S \end{array}$$

with f, \tilde{f} smooth. Then $\tilde{f}^* \circ p^* \circ p_* \xrightarrow{\tilde{f}^* \circ (\text{counit})} \tilde{f}^*$. Using the commutativity of the above diagram, this induces a map

$$f^* \circ p_* \rightarrow \tilde{p}^* \circ \tilde{f}^*.$$

Proposition 2.4.3. (Smooth base change) the map $f^* \circ p_* \rightarrow \tilde{p}^* \circ \tilde{f}^*$ is an equivalence of functors $(P)Sh(\text{Sm}_{\tilde{S}}) \rightarrow (P)Sh(\text{Sm}_T)$.

Proof. Recall $f^\# = f^*$. We have a commutative square

$$\begin{array}{ccc} \text{Sm}_{\tilde{T}} & \longrightarrow & \text{Sm}_{\tilde{S}} \\ \uparrow & & \\ \text{Sm}_T & \longrightarrow & \text{Sm}_S \end{array}$$

where, by the commutativity of the diagram we get $X \times_S \tilde{S} \simeq X \times_T \tilde{T}$. We see that the functors in this diagram are in fact

$$\begin{array}{ccc} \text{Sm}_{\tilde{T}} & \xrightarrow{\tilde{f}^\#} & \text{Sm}_{\tilde{S}} \\ \tilde{p}_* \uparrow & & \uparrow p_* \\ \text{Sm}_T & \xrightarrow{f^\#} & \text{Sm}_S \end{array}$$

□

Proposition 2.4.4. (Smooth projection formula) Consider $f : T \rightarrow S$ smooth, $E \rightarrow F \in Sh(\text{Sm}_S)$, and $G \in Sh(\text{Sm}_T)$ with $G \rightarrow f^*F$ (which corresponds to a map $f_\#G \rightarrow F$). Then, we have that

$$(f^*(E) \times_{f^*(F)} G) \simeq E \times_F f_\#(G).$$

We can show this on representable presheaves (exercise), but we cannot apply this everywhere, since the pullback on the left does not commute with colimits.

Proof. • Sheafification is left exact (i.e. commutes with finite limits) so we can reduce to the corresponding statement for presheaves.

- Colimits in PSh (more generally in ∞ -topoi) are universal, so we may replace G by a representable $X \in \text{Sm}_T$.
- For $U \in \text{Sm}_S$ we have that (note $f_{\#}^{\text{pre}}(X/T)(U) \simeq X/S$)

$$(E \times_F f_{\#}^{\text{pre}}(X/T))(U) \simeq E(U) \times_{F(U)} (X/S)(U/S) \simeq \coprod_{x \in (X/S)(U/S)} E(U) \times_{F(U)} \{x\}.$$

- In order to compute the left hand side, we use the following formula for computing $f_{\#}^{\text{pre}}$: take $H \in PSh(\text{Sm}_T)$. Take \mathcal{D} to be the opposite category of those schemes V which satisfy:

$$\begin{array}{ccc} V & \longleftarrow & U \\ sm \downarrow & & \downarrow \\ T & \xrightarrow{f} & S \end{array}$$

then

$$f_{\#}^{\text{pre}}(H)(U) \simeq \text{colim}_{\mathcal{D}} H(V)$$

so we have that

$$\begin{aligned} f_{\#}^{\text{pre}}(f^*(E) \times_{f^*(F)} X)(U) &= \text{colim}_{\mathcal{D}} E(V) \times_{F(V)} (X/T)(V/T) \\ &\simeq \text{colim}_{\mathcal{D}} \coprod_{x' \in (X/T)(V/T)} E(V) \times_{F(V)} \{x'\} \\ &\simeq \text{colim} \left\{ \begin{array}{ccc} V & \longleftarrow & U \\ x' \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array} \right\}^{\text{op}} E(V) \times_{F(V)} \{x'\} \simeq \coprod_{x \in (X/S)(U/S)} E(V) \times_{F(V)} \{x\} \end{aligned}$$

where the last isomorphism is by cofinality. □

2.5 f^* for essentially smooth maps

Consider $X \in \text{Sm}_S$, $x \in X$. Then the map

$$X_x^h = \text{Spec}(\mathcal{O}_{X,x}^h) \rightarrow S$$

is in general not smooth, but it is a limit of smooth maps $X_x^h \simeq \lim_{(U,u) \text{ Nis. neighbhd } U \text{ of } (X,x)} X_x^h$.

More generally, assume that $(X_i)_{i \in I}$ is a projective system in Sm_S with affine transition maps. Then $\tilde{X} = \lim_{i \in I} X_i$ exists.

Assume \tilde{X} is Noetherian and of finite dimension. Then our map $f : \tilde{X} \rightarrow S$ induces $f_{\text{pre}}^* : PSh(\text{Sm}_S) \rightarrow PSh(\text{Sm}_{\tilde{X}})$. This map f_{pre}^* has an explicit description as follows:

Take $Y \rightarrow \tilde{X}$ a smooth scheme. Then there exists $i_0 \in I$ and $Y_{i_0} \rightarrow X_{i_0}$ smooth such that

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & Y_{i_0} \\ \downarrow & \lrcorner & \downarrow \\ \tilde{X} & \longrightarrow & X_{i_0} \end{array}$$

Set Y_i equal to the pullback of $X_i \rightarrow X_{i_0} \leftarrow Y_{i_0}$, and we get $\lim_I Y_i \simeq \tilde{Y}$.

Lemma 2.5.1.

- (a) For $F \in PSh(\mathrm{Sm}_S)$, $f_{\mathrm{pre}}^*(F)(\tilde{Y}) \simeq \mathrm{colim}_I F(Y_i)$.
- (b) f_{pre}^* preserves sheaves, in particular $f^* = f_{\mathrm{pre}}^*|_{Sh}$.

In particular, $f_x : X_x^h \rightarrow S$, yields $f_{x,\mathrm{pre}}^*(F)(X_x^h) \simeq F_{(X,x)}$.

Proof. For (a), reduce to representables. For (b), check that for a sheaf F , $f_{\mathrm{pre}}^*(F)$ is excisive (and hence also a sheaf), since you can also approximate Nisnevich squares. \square

3 \mathbb{A}^1 -Invariance

Definition 3.0.1. A presheaf $F \in PSh(\mathrm{Sm}_S)$ is called \mathbb{A}^1 -invariant if $F(X) \xrightarrow{\mathrm{pr}^*} F(X \times \mathbb{A}^1)$ is an equivalence for all $X \in \mathrm{Sm}_S$.

We have that $PSh^{\mathbb{A}^1}(\mathrm{Sm}_S) \subseteq PSh(\mathrm{Sm}_S)$ is the full subcategory on \mathbb{A}^1 -invariant presheaves.

Definition 3.0.2. $Sh^{\mathbb{A}^1}(\mathrm{Sm}_S) := PSh^{\mathbb{A}^1}(\mathrm{Sm}_S) \cap Sh(\mathrm{Sm}_S)$ is the ∞ -category of *motivic spaces* (that is, this is the *unstable motivic homotopy category*).

Example 3.0.3.

- S is reduced, then \mathbb{G}_m is in $Sh^{\mathbb{A}^1}(\mathrm{Sm}_S)$
- S regular, then every smooth scheme also S is also regular, in particular $\Omega^\infty K(-) \in Sh^{\mathbb{A}^1}(\mathrm{Sm}_S)$, that is, algebraic K -theory.
- \mathbb{A}^1 is not \mathbb{A}^1 -invariant.

Note. F is \mathbb{A}^1 -invariant if and only if F is local with respect to the family of maps

$$\{X \times \mathbb{A}^1 \xrightarrow{\mathrm{pr}} X : X \in \mathrm{Sm}_S\}.$$

There exist Bousfield localizations

$$L_{\mathbb{A}^1} : PSh(\mathrm{Sm}_S) \rightleftarrows PSh^{\mathbb{A}^1}(\mathrm{Sm}_S) : \mathrm{incl}$$

$$L_{\mathrm{mot}} : PSh(\mathrm{Sm}_S) \rightleftarrows Sh^{\mathbb{A}^1}(\mathrm{Sm}_S) : \mathrm{incl}.$$

Explicit description: for any $n \in \mathbb{N}$, take $\Delta^n = \mathrm{Spec}(\mathbb{Z}[x_0, \dots, x_n]/(\sum x_i - 1))$ gives a cosimplicial scheme Δ^\bullet (with usual faces and degeneracies). Define a functor $\mathcal{H} : PSh(\mathrm{Sm}_S) \rightarrow PSh(\mathrm{Sm}_S)$ via

$$\mathcal{H}(F)(X) := \mathrm{colim}_{\Delta^{\mathrm{op}}} F(X \times \Delta^{\mathrm{op}})$$

which we claim is exactly the \mathbb{A}^1 localization $L_{\mathbb{A}^1}$. There is an obvious natural transformation $\alpha : \mathrm{id} \rightrightarrows \mathcal{H}$.

Proposition 3.0.4. We have an equivalence $\mathcal{H} \simeq L_{\mathbb{A}^1}$ (really we are looking at $i \circ L_{\mathbb{A}^1}$ so that both functors have codomain $PSh(\text{Sm}_S)$).

Proof. We will show below:

- (A) If $F \in PSh(\text{Sm}_S)$ then $\mathcal{H}(F)$ is \mathbb{A}^1 -invariant.
- (B) If $F \in PSh\mathbb{A}^1(\text{Sm}_S)$ then $F \xrightarrow{\alpha} \mathcal{H}(F)$ is an equivalence.

Note that (A) and (B) imply that, for any $F \in PSh(\text{Sm}_S)$, we have that $\mathcal{H}(F) \rightarrow \mathcal{H}(\mathcal{H}(F))$ has two natural transformations, which yield maps

$$H(\alpha_F), \alpha_{\mathcal{H}(F)} : \mathcal{H}(F) \rightarrow \mathcal{H}(\mathcal{H}(F)),$$

and we claim they are both equivalences.

We use the following proposition of Lurie:

Proposition 3.0.5. [Lur09, 5.2.7.4] Let \mathcal{C} be an ∞ -category, and let $L : \mathcal{C} \rightarrow \mathcal{C}$ be a functor with essential image $LC \subseteq \mathcal{C}$. Then the following are equivalent:

1. There exists a functor $f : \mathcal{C} \rightarrow \mathcal{D}$ with a fully faithful right adjoint $f \dashv g$ such that $g \circ f \simeq L$.
2. Regarded as a functor $\mathcal{C} \rightarrow LC$, we have that L is left adjoint to the inclusion $LC \subseteq \mathcal{C}$.
3. There is a natural transformation $\alpha : \text{id}_{\mathcal{C}} \rightrightarrows L$ such that, for all $c \in \mathcal{C}$, we have that

$$L(\alpha_c), \alpha_{Lc} : Lc \rightarrow LLc$$

are equivalent.

Therefore \mathcal{H} is the left adjoint of the inclusion of its essential image

$$\mathcal{H} : PSh(\text{Sm}_S) \rightleftarrows \text{im}(\mathcal{H}) : \text{incl}.$$

Moreover,

$$PSh^{\mathbb{A}^1}(\text{Sm}_S) \stackrel{(B)}{\subseteq} \text{im}(\mathcal{H}) \stackrel{(A)}{\simeq} PSh^{\mathbb{A}^1}(\text{Sm}_S),$$

therefore they are all equal.

(B) follows directly from $F(X) \xrightarrow{\simeq} F(X \times \Delta^n)$ for an \mathbb{A}^1 -invariant presheaf.

For (A), we want to prove that for any $F \in PSh(\text{Sm}_S)$ and $X \in \text{Sm}_S$,

$$\mathcal{H}(F)(X) \xrightarrow{\text{pr}^*} \mathcal{H}(F)(X \times \mathbb{A}^1)$$

is an equivalence. Let $\sigma : X \rightarrow X \times \mathbb{A}^1$ be the zero section, then $\sigma^* \circ \text{pr}^* \simeq \text{id}$. It remains to prove that $\text{pr}^* \circ \sigma^* \simeq \text{id}$ on $\mathcal{H}(F)(X \times \mathbb{A}^1)$. So we will construct an explicit simplicial homotopy.

If \mathcal{C} is any ∞ -category, we may talk about simplicial objects in it as $s\mathcal{C} = \text{Fun}(\Delta^{\text{op}}, \mathcal{C})$, which consists of “simplicial objects in \mathcal{C} ,” which we denote X_{\bullet} , Y_{\bullet} , etc. To define a simplicial homotopy, we can define

$$\begin{array}{ccc} \Delta & \begin{array}{c} \xrightarrow{i_0} \\ \xrightarrow{i_1} \end{array} & \Delta/[1] \xrightarrow{\delta} \Delta \\ [n] & \mapsto & ([n] \mapsto * \begin{array}{c} \xrightarrow{0} \\ \xrightarrow{1} \end{array} [1]) \mapsto [n] \end{array}$$

Definition 3.0.6. A *simplicial homotopy* between $f, g : X_\bullet \rightarrow Y_\bullet$ is a map $h : \delta^*(X_\bullet) \rightarrow \delta^*(Y_\bullet)$ in $\text{Fun}((\Delta/[1])^{\text{op}}, \mathcal{C})$ such that $i_0^*(h) = f$ and $i_1^*(h) = g$.

Note. Any functor $\mathcal{C} \rightarrow \mathcal{D}$ preserves simplicial homotopies.

Lemma 3.0.7. Assume \mathcal{C} is cocomplete. If $f, g : X_\bullet \rightarrow Y_\bullet$ in $s\mathcal{C}$ are simplicially homotopic, then the induced maps $|f|, |g| : |X_\bullet| \rightarrow |Y_\bullet|$ are homotopic in \mathcal{C} .

The reason is that

$$\text{colim}_{(\Delta/[1])^{\text{op}}} \delta^*(X_\bullet) \xrightarrow{\delta} |X_\bullet| \longrightarrow \text{colim}_{\Delta^{\text{op}}} X_\bullet$$

$\begin{array}{c} \xleftarrow{i_1} \\ \xleftarrow{i_0} \end{array}$

we may show δ is an equivalence.

Claim: For $X \in \text{Sm}_S$, the maps

$$X \times \mathbb{A}^1 \times \Delta^\bullet \xrightarrow[\sigma_{\text{opr}}]{\text{id}} X \times \mathbb{A}^1 \times \Delta^\bullet$$

are simplicially homotopic.

Proof. Write t coordinate on \mathbb{A}^1 . Then take

$$h : \delta^*(X \times \mathbb{A}^1 \times \Delta^\bullet) \rightarrow \delta^*(X \times \mathbb{A}^1 \times \Delta^\bullet)$$

where $([n] \xrightarrow{\pi} [1])$ maps to $h(\pi) : X \times \mathbb{A}^1 \times \Delta^n \rightarrow X \times \mathbb{A}^1 \times \Delta^n$, and we have that

$$h(\pi) : t \mapsto \left(\sum_{j \in \pi^{-1}(1)} x_j \right) \cdot t.$$

□

Thus $i_0^*(h)$ is given by $t \mapsto 0$ and $i_1^*(h)$ is given by $t \mapsto t$.

□

Example 3.0.8. $L_{\mathbb{A}^1}$ does not preserve sheaves, nor does it preserve \mathbb{A}^1 -invariant objects.

An explicit counter example is given in [MV99, Ex. 3.2.7]. Let $S = \text{Spec}(k)$, where k is a field. Then Let $U_0 = \mathbb{A}^1 \setminus \{0\}$, and $U_1 = \mathbb{A}^1 \setminus \{1\}$. Then the intersection $U_{01} = U_0 \cap U_1$ is \mathbb{A}^1 -invariant, since we may see that σ^* is an isomorphism in the following diagram:

$$\begin{array}{ccc} U_{01}(X) & \xleftarrow{\sigma^*} & U_{01}(X \times \mathbb{A}^1) \\ \downarrow & & \downarrow \\ U_0(X) & \xleftarrow{\simeq} & U_0(X \times \mathbb{A}^1) \end{array}$$

Choose a closed immersion $U_{01} \xrightarrow{cl} \mathbb{A}^1$, and construct a non-smooth scheme

$$F = (U_0 \times \mathbb{A}^n) \coprod_{U_{01}} (U_1 \times \mathbb{A}^n)$$

If $X \in \text{Sm}_S$ is connected, then

$$\text{Hom}(X, F) = \text{Hom}(X, U_0 \times \mathbb{A}^n) \coprod_{\text{Hom}(X, U_{01})} \text{Hom}(X, U_1 \times \mathbb{A}^n).$$

That is, $F(X) \simeq (U_0 \times \mathbb{A}^n) \coprod_{U_{01}}^{\text{pre}} (U_1 \times \mathbb{A}^n)(X)$. Now we identify $L_{\mathbb{A}^1}F(X)$ with $\mathcal{H}F(X)$, and we may explicitly give this as

$$L_{\mathbb{A}^1}F(X) \simeq L_{\mathbb{A}^1}(U_0 \times \mathbb{A}^n) \coprod_{U_{01}}^{\text{pre}} L_{\mathbb{A}^1}(U_1 \times \mathbb{A}^n)(X) \simeq (U_0 \coprod_{U_{01}}^{\text{pre}} U_1)(X)$$

since $L_{\mathbb{A}^1}$ commutes with coproducts. However we claim that this thing we ended with is not a sheaf. If it were, then it would be equivalent to its sheafification, that is, since sheafification commutes with coproducts,

$$U_0 \coprod_{U_{01}}^{\text{pre}} U_1 \simeq U_0 \coprod_{U_{01}} U_1 \simeq \mathbb{A}^1.$$

But \mathbb{A}^1 was not \mathbb{A}^1 -invariant.

(Another example would be $\mathbb{A}^1 \coprod_0 \mathbb{A}^1$, which looks like the x-axis union the y-axis, which is singular.)

Proposition 3.0.9. We have an equivalence

$$L_{\text{mot}} \simeq \text{colim}^{\text{pre}} (\text{id} \rightarrow L \circ L_{\mathbb{A}^1} \rightarrow (L \circ L_{\mathbb{A}^1})^2 \rightarrow \dots)$$

Proof. Denote the right hand side by \mathcal{H}' .

- For $F \in \text{PSh}(\text{Sm}_S)$, we have that $\mathcal{H}'(F)$ is a sheaf. (Can check on Nisnevich squares, and use that filtered colimits of pullback squares in spaces is again a pullback square.
- $\mathcal{H}'(F)$ is \mathbb{A}^1 -invariant). This is because we can rewrite:

$$\mathcal{H}' \simeq \text{colim}^{\text{pre}} (\text{id} \rightarrow L_{\mathbb{A}^1} \rightarrow L_{\mathbb{A}^1} \circ (L \circ L_{\mathbb{A}^1}) \rightarrow \dots)$$

which stays \mathbb{A}^1 -invariant by a cofinality argument.

To check left adjointness, we have to show that $\text{Map}(\mathcal{H}'(F), E) \simeq \text{Map}(F, E)$. But by the formula, the left hand side is just

$$\text{Map}(\mathcal{H}'(F), E) \simeq \lim_{\mathbb{N}} \text{Map}((L \circ L_{\mathbb{A}^1})^n(F), E) \simeq \lim_{\mathbb{N}} \text{Map}(F, E) \simeq \text{Map}(F, E)$$

since E is \mathbb{A}^1 -invariant and a sheaf. □

4 \mathbb{A}^1 -invariance and base change functors

Consider a map $f : T \rightarrow S$ of base schemes, then

$$f_* : PSh(\mathrm{Sm}_T) \rightarrow PSh(\mathrm{Sm}_S)$$

preserves \mathbb{A}^1 -invariance, so we get induced adjunctions

$$f_{\mathbb{A}^1}^* := L_{\mathbb{A}^1} \circ f_{\mathrm{pre}}^* \circ \mathrm{incl} : PSh^{\mathbb{A}^1}(\mathrm{Sm}_S) \rightleftarrows PSh^{\mathbb{A}^1}(\mathrm{Sm}_T) : f_*$$

and similarly on the level of sheaves

$$f_{\mathrm{mot}}^* := L_{\mathrm{mot}} \circ f_{\mathrm{pre}}^* : Sh^{\mathbb{A}^1}(\mathrm{Sm}_S) \rightleftarrows Sh^{\mathbb{A}^1}(\mathrm{Sm}_T) : f_*$$

Proposition 4.0.1. If $i : Z \xrightarrow{\mathrm{cl}} S$ is a closed immersion, then

$$L_{\mathrm{mot}} \circ i_*(F) \rightarrow i_* \circ L_{\mathrm{mot}}(F)$$

is an equivalence for any $F \in PSh(\mathrm{Sm}_Z)$ if $F(\emptyset) = *$.

Proof. (1)

$$\begin{aligned} (L_{\mathbb{A}^1} \circ i_*)(F) &\simeq \mathrm{colim}_{\Delta_{\mathrm{op}}^{\mathrm{pre}}} (i_* F)(-\times \Delta^\bullet) \simeq i_* \mathrm{colim}_{\Delta_{\mathrm{op}}^{\mathrm{pre}}} F(-\times \Delta^\bullet) \\ &\simeq i_* L_{\mathbb{A}^1} F \end{aligned}$$

(2) We have that

$$L \circ L_{\mathbb{A}^1} \circ i_*(F) \simeq L \circ i_* \circ L_{\mathbb{A}^1}(F) \simeq i_* L \circ L_{\mathbb{A}^1}(F)$$

where the second isomorphism is by Lemma 2.4.1. Additionally, we have that

$$L_{\mathrm{mot}} \circ i_* \circ F = \mathrm{colim}_{\mathbb{N}}^{\mathrm{pre}} (L \circ L_{\mathbb{A}^1})^n \circ i_*(F).$$

Since i_* commutes with all presheaf colimits, and using 1), we obtain that this is isomorphic to $i_* \circ L_{\mathrm{mot}}(F)$. □

Proposition 4.0.2. Analogously to Proposition 2.4.2, if $Z \xrightarrow{\mathrm{cl}} S$ is a closed immersion, then

$$i_* : Sh^{\mathbb{A}^1}(\mathrm{Sm}_Z) \rightarrow Sh^{\mathbb{A}^1}(\mathrm{Sm}_S)$$

commutes with weakly contractible colimits. The proof is completely analogous.

4.1 Smooth maps

If $f : T \rightarrow S$ is smooth, then $f^* \simeq f_{\mathrm{pre}}^*$, since the map is given by restriction. Moreover, the map $f_{\mathrm{pre}}^* \simeq f^\# : PSh(\mathrm{Sm}_S) \rightarrow PSh(\mathrm{Sm}_T)$ preserves \mathbb{A}^1 -invariant (pre)sheaves, so we get an adjunctions:

$$f_\# \dashv f^* = f^\# \dashv f_*.$$

Additionally, we get

$$f_{\#}^{\mathrm{mot}} := L_{\mathrm{mot}} \circ f_{\#}^{\mathrm{pre}} \circ \mathrm{incl} : Sh^{\mathbb{A}^1}(\mathrm{Sm}_T) \rightleftarrows Sh^{\mathbb{A}^1}(\mathrm{Sm}_S) : f^* = f^\#.$$

In particular, we also have

$$f_{\text{mot}}^* \simeq f^* \simeq f_{\text{pre}}^*$$

for smooth maps. And additionally,

$$L_{\text{mot}} \circ f^* \xrightarrow{\sim} f^* \circ L_{\text{mot}}.$$

To see this, we look at the series of adjunctions

$$\begin{array}{ccc} PSh(\text{Sm}_T) & \xrightarrow{f_*} & PSh(\text{Sm}_S) \\ \text{red } L_{\text{mot}} \downarrow & \uparrow \text{incl} & \text{red } L_{\text{mot}} \downarrow \\ & \text{red } f^* & \\ & \uparrow \text{incl} & \\ Sh^{\mathbb{A}^1}(\text{Sm}_T) & \xrightarrow{f_*} & Sh^{\mathbb{A}^1}(\text{Sm}_S) \end{array}$$

and note the commutativity of the red arrows.

4.2 Stalkwise detection of motivic equivalences

Definition 4.2.1. A map $E \xrightarrow{\varphi} F$ in $PSh(\text{Sm}_S)$ is called a *motivic equivalence* if $L_{\text{mot}}(\varphi)$ is an equivalence.

Recall: For $X \in \text{Sm}_S$, $x \in X$, $f_* : X_x^h \rightarrow S$ essentially smooth, we saw that $f_{x,\text{pre}}^*$ preserves sheaves, therefore we write f_x^* for $f_{x,\text{pre}}^*$.

Lemma 4.2.2. A map $E \xrightarrow{\varphi} F$ in $PSh(\text{Sm}_S)$ is a motivic equivalence iff $\forall X \in \text{Sm}_S$ and $\forall x \in S$, we have that $f_x^*(\varphi)$ is a motivic equivalence in $PSh(\text{Sm}_{X_x^h})$.

Proof. As for smooth maps, f_x^* preserves \mathbb{A}^1 -invariant sheaves. This implies that $L_{\text{mot}} \circ f_x^* \xrightarrow{\sim} f_x^* \circ L_{\text{mot}}$.

\Rightarrow : We have that φ is a motivic equivalence implies $L_{\text{mot}}(\varphi)$ is a motivic equivalence, which gives that $f_x^* L_{\text{mot}}(\varphi) = L_{\text{mot}} \circ f_x^*(\varphi)$ is a motivic equivalence. Therefore $f_x^*(\varphi)$ is a motivic equivalence.

\Leftarrow : Let $f_x^*(\varphi)$ motivic equivalence for all $X \in \text{Sm}_S$ and for all $x \in X$. This implies that $L_{\text{mot}} \circ f_x^* = f_x^* \circ L_{\text{mot}}(\varphi)$ is a motivic equivalence. This is now a map of presheaves over our scheme, so we may take global sections to see that $f_x^* \circ L_{\text{mot}}(\varphi)(X_x^h)$ is an equivalence.

This is exactly $L_{\text{mot}}(\varphi)_{X,x}$, so we see that $L_{\text{mot}}(\varphi)$ is a stalkwise equivalence between sheaves, and therefore an equivalence. \square

We have an even stronger version of the lemma as follows.

Lemma 4.2.3. A map $E \xrightarrow{\varphi} F$ in $PSh(\text{Sm}_S)$ is a motivic equivalence if and only if $f_s^*(\varphi)$ is a motivic equivalence in $PSh(\text{Sm}_{S_s^h})$ for $s \in S$.

The reason is that in the diagram

$$\begin{array}{ccccc} X_x^h & \longrightarrow & X \times_S S_s^h & \longrightarrow & X \\ & \searrow g & \downarrow sm & & \downarrow sm \\ & & S_s^h & \xrightarrow{f_s} & S \end{array}$$

where g is essentially smooth, then we have that $L_{\text{mot}} \circ g^* \simeq g^* \circ L_{\text{mot}}$.

4.3 Gluing

Let $Z \xrightarrow{i} S \xleftarrow{o} U := S \setminus Z$, where i is a closed immersion. We want to show that, for $E \in \text{Sh}^{\mathbb{A}^1}(\text{Sm}_S)$, the square

$$\begin{array}{ccc} j_{\#}^{\text{mot}} j^* E & \longrightarrow & E \\ \downarrow & & \downarrow \\ j_{\#}^{\text{mot}}(*) & \longrightarrow & i_* i_{\text{mot}}^* E \end{array}$$

is a pushout square.

First we want to reduce to the case where E is representable.

Lemma 4.3.1. The ∞ -cat $\text{Sh}(\text{Sm}_S)$ is generated by representables under weakly contractible colimits.

Proof. Let $E \in \text{Sh}(\text{Sm}_S)$, then we may write it as a colimit of representables $E \simeq \text{colim}_{y(X) \rightarrow E} y(X)$, and we have that $y(-)/E$ has an initial object $y(\emptyset)$. Moreover, we have seen that all functors involved commute with weakly contractible colimits.

Now it suffices to show that $L_{\text{mot}}(-)$ applied to the diagram of presheaves

$$\begin{array}{ccc} j_{\#}^{\text{mot}} j^* X & \longrightarrow & E \\ \downarrow & & \downarrow \\ U/S & \longrightarrow & i_* i_{\text{mot}}^* X \end{array}$$

is a pushout.

Note that L_{mot} commutes with pushouts, and that for formal reasons, we have seen that the following hold:

$$\begin{aligned} L_{\text{mot}} \circ j_{\#} &\simeq j_{\#}^{\text{mot}} \circ L_{\text{mot}} \\ L_{\text{mot}} \circ i^* &\simeq i_{\text{mot}}^* \circ L_{\text{mot}}. \end{aligned}$$

Also we have that L_{mot} commutes with j^* as j is smooth, and L_{mot} commutes with i_* (for \dots F with $F(\emptyset) = *$).

So it remains to show that $X \coprod_{X_U} U \rightarrow i_* X_Z$ is a motivic equivalence in $\text{PSh}(\text{Sm}_S)$ (where $X_U = X \times_S U \rightarrow S$ and $X_Z := X \times_S Z \rightarrow Z$).

Lemma 4.3.2. A morphism $E \rightarrow F$ in $\text{PSh}(\text{Sm}_S)$ is a motivic equivalence if and only if for all $Y \in \text{Sm}_S$, and maps $\emptyset : Y \rightarrow F$,

$$\begin{array}{ccc} Y \times_F E & \longrightarrow & E \\ (*) \downarrow & & \downarrow \\ Y & \xrightarrow{\emptyset} & F \end{array}$$

the induced map $(*)$ is a motivic equivalence.

Proof. Let $F \simeq \text{colim}_{Y \rightarrow F} Y$, then

$$\begin{array}{ccc} \text{colim}_{Y \rightarrow F} (Y \times_F E) & \xrightarrow{\simeq} & (\text{colim}_{Y \rightarrow F} Y) \times_F E \xrightarrow{\simeq} F \times_F E \simeq E \\ \downarrow & & \downarrow \\ \text{colim}_{Y \rightarrow F} Y & \xrightarrow{\simeq} & F \end{array}$$

and apply $L_{\text{mot}}(-)$. □

Hence we must show that, for all $\emptyset : Y \rightarrow i_* X_Z$ in $\text{PSh}(\text{Sm}_S)$, we have that the map

$$(X \coprod_{X_U} U) \times_{i_* X_Z} Y \rightarrow Y$$

is a motivic equivalence.

We want to reduce to the case $Y = S$. Consider the diagram of pullback squares

$$\begin{array}{ccccc} Z' & \xleftarrow{cl} & Y & \longleftarrow & U' \\ p' \downarrow & & \downarrow p & & \downarrow p'' \\ Z & \xrightarrow{i, cl} & S & \xleftarrow{j, o} & U \end{array}$$

which gives

$$(X \coprod_{X_U} U) \times_{i_* X_Z} p_{\#}(Y) \rightarrow p_{\#}(Y)$$

which is equal to

$$p_{\#}(p^*(X \coprod_{X_U} U) \times_{p^* i_* X_Z} Y \rightarrow Y) = p_{\#}(p^* X \coprod_{(p^* X)_{U'}} U' \times_{i'_*(p^* X)_{Z'}} Y \rightarrow Y).$$

Smoothness implies that

$$p^* i_* X_Z \simeq i'_* p'^* X_Z \simeq i'_* p'^* i^* X \simeq i'_* i'^*(p^* X) \simeq i'_*(p^* X)_{Z'}$$

so we can reduce to $S = Y$ in $(*)$.

Now we want to reduce to S being Henselian local. Consider, for all $s \in S$, the map $f : \text{Spec}(\mathcal{O}_{S,s}^h) \rightarrow S$. Then it suffices, by the lemma, to show that $f^*(*)$ is a motivic equivalence. This is essentially a “smooth base change.”

So we may assume $Y = S$ is the spectrum of a Henselian local ring. Note that, by our adjunctions, we have bijections between the sets of morphisms

$$\{\emptyset : Y \rightarrow i_* X_Z\} \leftrightarrow \left\{ \begin{array}{ccc} Z & \longrightarrow & X_Z \\ & \searrow & \downarrow \\ & & Z \end{array} \right\} \leftrightarrow \left\{ \begin{array}{ccc} Z & \xrightarrow{s} & X \\ & \searrow & \downarrow \\ & & S \end{array} \right\}$$

Consider the map

$$\begin{aligned} \Psi : (\mathrm{Sm}_S, \text{ plus sections}) &\rightarrow \mathrm{Fun}(\mathrm{Sm}_S^{\mathrm{op}}, \mathrm{Set}) \hookrightarrow \mathrm{PSh}(\mathrm{Sm}_S) \\ (X, s) &\longmapsto \Psi(X, s) := (X \coprod_{X_U} U) \times_{i_* X_Z} S \xrightarrow{\varphi(X, s)} S \end{aligned}$$

We now need two lemmas to finish the proof

Lemma 4.3.3. For $(X, s) \rightarrow (X', s')$ with $X \rightarrow X'$ étale, we have that the map $\Psi(X, s) \rightarrow \Psi(X', s')$ is a local equivalence.

Lemma 4.3.4. If $s'' : Z \hookrightarrow \mathbb{A}_Z^m \rightarrow \mathbb{A}_S^m$ is the “zero section,” then $\varphi(\mathbb{A}_S^m, s'')$ is a motivic equivalence.

Suppose we had these two lemmas, and let $x \in \mathrm{Spec}(A/I) = Z$ closed point, and $S = \mathrm{Spec}(A)$. Then

$$\begin{array}{ccc} & & U \\ & \nearrow s' & \downarrow \text{ét} \\ & X & \mathbb{A}_S^m \\ \nearrow s & \downarrow sm & \nearrow \\ Z & \xrightarrow{cl} & S \end{array}$$

$s(x) \in X$, $s(Z) \subseteq U$, $s^{-1}(U) \supseteq D(a) \ni x$, for $a \notin X$, so a unit, $Z = D(a)$.

Without loss of generality, $s'' : Z \rightarrow \mathbb{A}_S^m$ is the zero section, and we get

$$(X, s) \leftarrow (U, s') \rightarrow (\mathbb{A}_S^m, s'')$$

applying Lemma 4.3.3, since these maps are étale, and applying Lemma 4.3.4, we get the equivalences

$$\begin{array}{ccc} L(\Psi(X, s)) & \xleftarrow{\simeq} & L(\Psi(U, s')) & \xrightarrow{\simeq} & L(\Psi(\mathbb{A}_S^m, s'')) \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array} \quad \varphi(\mathbb{A}_S^m, s'')$$

and we are done. □

Proof of Lemma 4.3.3: Prove it “stalkwise.”

Let $Y \in \mathrm{Sm}_S$. Then

$$\Psi(X, s)(Y) = ((X \coprod_{X_U} U) \times_{i_* X_Z} S)(Y) = \begin{cases} * & Y_Z = \emptyset \\ \mathrm{Hom}_S(Y, X) \times_{\mathrm{Hom}_Z(Y_Z, X_Z)} * & Y_Z \neq \emptyset \end{cases}$$

where the inclusion of the point is given by $Y_Z \rightarrow Z \xrightarrow{\emptyset} X_Z$.

If $Y_Z = \emptyset$, then Y lives completely over U (that is, $Y = Y_U$), and we have that

$$\begin{array}{ccc} X_U(Y) & \longrightarrow & U(Y) \simeq * \\ \parallel & & \downarrow \\ X(U) & \longrightarrow & (X \coprod_{X_U} U)(Y) \simeq * \end{array}$$

then $(i_* X_Z)(U) = X_Z(Y_Z) = X_Z(\emptyset) = *$, and therefore $S(Y) = *$.

If $Y_Z \neq \emptyset$ then we get

$$\begin{array}{ccc} X_U(Y) = \emptyset & \longrightarrow & U(Y) = \emptyset \\ \downarrow & & \downarrow \\ X(Y) & \xlongequal{\quad} & (X \cup_{X_U} U)(Y) \end{array}$$

If $Y_Z \neq \emptyset$ look at the following diagram in $\Psi(X', s')(Y)$:

$$\begin{array}{ccc} Y & \xrightarrow{f'} & X' \\ \text{cl} \uparrow & & \uparrow \text{cl} \\ Y_Z & \longrightarrow & Z \xrightarrow{\vartheta'} X'_Z \end{array}$$

where

$$\begin{array}{ccc} Z \xrightarrow{s} X & & Z \xrightarrow{\vartheta} X_Z \\ \downarrow s' \swarrow & & \downarrow \vartheta' \swarrow \\ X' & & X'_Z \end{array}$$

and we want to find a unique preimage in $\Psi(X, s)(Y)$. So we have a diagram

$$\begin{array}{ccccccc} Y_Z & \longrightarrow & Z & \xrightarrow{\vartheta} & X_Z & \hookrightarrow & X \\ & & \downarrow \vartheta' & \swarrow & \downarrow \vartheta' & & \downarrow et \\ & & X'_Z & \xrightarrow{f} & X & & \\ & \nearrow & & & & & \\ Y & \xrightarrow{f'} & X' & & & & \end{array}$$

As Y is Henselian, $Y_Z \neq \emptyset$, there exists a lift f , indicated in green, such that $f \in \text{Hom}_S(Y, X)$. This is actually a map in $\Psi(X, s)(Y)$ by construction.

As Y is connected, $X \rightarrow X'$ unramified, this f is unique [GR71, I, Cor. 5.4], if $X \rightarrow X'$ were separated. We can extend our diagram

$$\begin{array}{ccccccccc}
Y_Z & \longrightarrow & Z & \xrightarrow{\varnothing} & X_Z & \hookrightarrow & X & \longleftarrow & \bar{U} & \xleftarrow{o} & U \\
\downarrow & & \downarrow \varnothing' & \swarrow & \downarrow & & \downarrow et & & \downarrow et & & \\
Y & \longrightarrow & X'_Z & \xrightarrow{f} & X & \longrightarrow & X' & \longleftarrow & U' & \xleftarrow{o} & \\
& & \downarrow & \swarrow & \downarrow & & \downarrow & & \downarrow & & \\
& & Y & \xrightarrow{f'} & X' & & & & & &
\end{array}$$

Without loss of generality, we may restrict to X' affine, and may restrict to X affine. Therefore we are done. \square

Proof of Lemma 4.3.4: Let $Y \in \text{Sm}_S$. Again the case $Y_Z = \varnothing$ is trivial. So we construct a homotopy

$$(\Psi(\mathbb{A}_S^m, s'') \times \mathbb{A}_S^1)(Y) \rightarrow \Psi(\mathbb{A}_S^m, s'')(Y)$$

via

$$\begin{array}{ccccc}
Y & \longrightarrow & \Psi(\mathbb{A}_S^m, s'') \times \mathbb{A}_S^1 & \xrightarrow{(*)} & \Psi(\mathbb{A}_S^m, s'') \\
\uparrow & & \downarrow \subseteq & & \\
& & \mathbb{A}_S^m \times \mathbb{A}_S^1 & \longrightarrow & \mathbb{A}_S^m \\
& & \uparrow \text{id} \times \{t\} & & \\
Y_Z & \longrightarrow & Z & \xrightarrow{0} & \mathbb{A}_S^m
\end{array}$$

Where the map $(*)$ is given by $(x_1, \dots, x_m, t) \mapsto (x_1 t, \dots, x_m t)$. If $t = 1$ this is the identity, and if $t = 0$ this is S . \square

4.4 Brief sketch of Cisinski's proof of cdh-descent for KH

This is a pointed version of $Sh_*^{\mathbb{A}^1}(\text{Sm}_S) =: \mathcal{H}_*(S)$. $f_{\#} \dashv f^* \dashv f_*$.

The gluing theorem implies here, that

$$j_{\#} j^* E \rightarrow E \rightarrow i_* i^* E$$

is a cofiber sequence (here we ignored the “mot”)

$(\mathbb{P}^1, \infty) \in \mathcal{H}_*(S)$. Note that $\mathcal{H}_*(S)$ is a symmetric monoidal category with respect to \wedge .

There is a “universal construction”

$$\Sigma_{\mathbb{P}^1}^{\infty} : \mathcal{H}_*(S) \rightarrow \mathcal{SH}(S)$$

which gives us *stable motivic homotopy theory*.

Note that the sequence above is also a cofiber=fiber sequence in $\mathcal{SH}(S)$.

Lemma 4.4.1. The unit $\text{id} \xrightarrow{\eta} j^* j_{\#}$ is an equivalence.

Proof.

$$\begin{array}{ccc} V & \xlongequal{\quad} & V \\ \downarrow & \lrcorner & \downarrow \\ U & \longrightarrow & S \end{array}$$

□

Corollary 4.4.2. $j_{\#}$ is fully faithful.

Proof. $\text{Map}(j_{\#}A, j_{\#}B) \simeq \text{Map}(A, j^*j_{\#}B) \simeq \text{Map}(A, B)$.

□

Lemma 4.4.3. The pair $(j^*, i^*) : \mathcal{SH}(S) \rightarrow \mathcal{SH}(U) \times \mathcal{SH}(Z)$ reflects equivalences.

Proof. Use the cofiber sequence

$$\begin{array}{ccccc} j_{\#}j^*E & \longrightarrow & E & \longrightarrow & i_*i^*E \\ \downarrow \simeq & & \downarrow & & \downarrow \simeq \\ j_{\#}j^*F & \longrightarrow & F & \longrightarrow & i_*i^*F \end{array}$$

□

Exercise 4.4.4.

1. $i^*i_* \rightarrow \text{id}$ is an equivalence, and hence i_* is fully faithful.
2. $\mathcal{SH}(S) \simeq \mathcal{SH}(S_{\text{red}})$ (for this we use $\emptyset \xrightarrow{o} S \xleftarrow{cl} S_{\text{red}}$).

Theorem 4.4.5. (Ayoub, Proper Base Change Theorem) For the diagram of the form

$$\begin{array}{ccc} X' & \xleftarrow{i'} & Y' \\ p \downarrow & \lrcorner & \downarrow q \\ X & \xleftarrow{i} & Y \end{array}$$

where p is proper, then we obtain $i^*p_* \simeq q_*i'^*$.

For the Smooth Base Change Theorem, we let p be anything, and require i to be smooth.

An abstract blowup square is

$$\begin{array}{ccccc} X' \setminus Y' & \xlongequal{\quad} & U' & \xrightarrow{o} & X' & \xleftarrow{i'} & Y' \\ & & \downarrow \simeq & & \downarrow \text{proper} & & \downarrow q \\ X \setminus Y & \xlongequal{\quad} & U & \xrightarrow{o} & X & \xleftarrow{i} & Y \end{array}$$

The *cdh topology* is the topology given by $\{p, i\}$ of all abstract blowup squares and the Nisnevich topology.

Proposition 4.4.6. Given an a.b.s. as above, where ℓ denotes the composite $Y' \rightarrow X$, we have that

$$\begin{array}{ccc} E & \longrightarrow & p_*p^*E \\ \downarrow & & \downarrow \\ i_*i^*E & \longrightarrow & \ell_*\ell^*E \end{array}$$

is a cartesian square in $\mathcal{SH}(S)$.

Proof. Test this after applying j^*, i^* . For the first one, we get

$$\begin{array}{ccc} j^*E & \longrightarrow & j^*p_*p^*E \\ \downarrow & & \downarrow \\ j^*i_*i^*E & \longrightarrow & j^*\ell_*\ell^*E \end{array}$$

Recall our cofiber sequence remains a cofiber sequence after applying $j^*(-)$. However recall that $j^*j_\# \simeq \text{id}$, so we have that

$$j_*j_\#j^*E \simeq j^*E \xrightarrow{\simeq} j_*E \rightarrow j_*i_*i^*E \simeq 0.$$

By SBC, we get that the top right corner is

$$j^*p_*p^*E \simeq j'^*p^*E \simeq j^*E,$$

and therefore the top map is the identity. Finally, we also get that the bottom right corner is

$$j^*\ell_*\ell^*E \simeq j^*i_*q_*i^*q^*E$$

$$0 \simeq j_\#j^*i_*(\dots) \rightarrow i_*(\dots) \xrightarrow{\simeq} i_*(i^*i_*)(\dots)$$

therefore we get that the bottom row is all $\simeq 0$.

After applying i^* , we get

$$\begin{array}{ccc} i^*E & \longrightarrow & i^*p_*p^*E \\ \downarrow & & \downarrow \\ i^*i_*i^*E & \longrightarrow & i^*\ell_*\ell^*E \end{array}$$

we first see the bottom left is $i^*i_*i^*E \simeq i^*E$, so the left map is an equivalence. Since the p is proper, we may use PBC to see that the upper right corner is

$$i^*p_*p^*E \simeq q_*i'^*p^*E$$

but the bottom corner is

$$i^*\ell_*\ell^*E \simeq i^*(iq)_*(p')^*E \simeq (i^*i_*)q_*i'^*p^*E \simeq q_*i'^*p^*E$$

so the map along the right is also an equivalence. □

This implies, for $E = KH \in \mathcal{SH}(S)$, we have that $f^*KH \simeq KH$ (cartesian object).

$\text{BGL} \times \mathbb{Z}$ “specific model.”

Evaluate on X

$$\begin{array}{ccc} KH(X) & \longrightarrow & KH(X') \\ \downarrow & & \downarrow \\ KH(Z) & \longrightarrow & KH(Y') \end{array} \quad \lrcorner$$

this shows that homotopy K -theory KH satisfies cdh-descent. For details, see Cisinski’s article [Cis13].

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