SETS, GROUPS, AND GEOMETRY
Spring 2025

THOMAS BRAZELTON

ABSTRACT. Course notes for MATH101: Sets, groups and geometry, taught at Harvard in Spring 2025.

1. SETS

A *set* is a collection of things, and these things are called elements. We won't give a formal definition of a set, since this gets us too deep into mathematical logic, so we'll kind of take a set as a given and build mathematics on top of it.

We denote by $\{1, 2, 3\}$ the set whose elements are the numbers 1, 2, and 3. These curly braces are used to list the elements of a set.

Example 1.1. The set

$$S = \{a, b, c, d\}$$

is a set consisting of four elements, which are *letters* a, b, c, and d.

Note 1.2. Elements are not allowed to be repeated! For instance, $\{a, b, a, c, d\}$ is not a valid set.[1]

Notation 1.3. We use the symbol $\in$ to denote if an element is in a set. So if $T = \{0, 4, 1, 6\}$, we might write

$$1 \in T$$

to mean that 1 is an element of T. We will write $\notin$ to say something is **not** an element of a set. So for instance

$$2 \notin T.$$

Example 1.4. We denote by $\mathbb{N}$ the set of all *natural numbers*, meaning counting numbers including zero:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}.$$

We denote by $\mathbb{Z}$ the set of all *integers*[2] meaning all positive and negative counting numbers:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

We denote by $\mathbb{Q}$ the set of all *rational numbers*, meaning numbers of the form $\frac{p}{q}$ where p and q are integers, and $q \neq 0$.

Example 1.5. We don't just need to have numbers and letters be elements of sets. We can really let *anything* be an element in a set. For instance

$$S = \{\bigcirc, \triangle, \square\}.$$

[1] This is a convention that we're not allowing for repeated elements. We can build a different type of set theory where you *can* have repeated elements in sets, these are called *multisets.* The math that you build with these becomes a lot more complicated though.

[2] This letter comes from the German *Zahlen*, meaning "numbers."

We can also have *sets* being elements of sets. For instance we can take

$$B = \{\mathbb{N}, \mathbb{Z}, 3, \{4\}\}.$$

This is a set with four elements – the set of natural numbers, the set of integers, the number 3, and the set with one element which is the number 4. This might feel weird but we'll get used to it soon enough.

In the above examples, we didn't list out every element of a set when we wrote it, instead we did a ... when the pattern became clear. For instance what is the following set:

$$A = \{0, 3, 6, 9, 12, 15, \ldots\}.$$

It is the set of all multiples of three! Instead of listing it out, we might *build it*, meaning give a rule for elements to be a part of it. This is done using set builder notation:

$$A = \{3n \colon n \in \mathbb{N}\}.$$

This means A is the set of all numbers of the form $3n$ where n is an element of $\mathbb{N}$.[3]

A special set is the *empty set*, which has no elements. We could write it as $\{\}$ if we wanted, but we use special notation for it, namely $\varnothing$.

1.1. **Cardinality.** If A is a set, we denote by $|A|$ the *cardinality* of the set, roughly meaning its size. It is the number of elements in the set, possibly infinite.

Example 1.6. The cardinality of some sets we've discussed are:

$$\begin{aligned} |\{a, b, c, d\}| &= 4 \\ |\mathbb{N}| &= \infty \\ |\mathbb{Z}| &= \infty \\ |\{\bigcirc, \triangle, \square\}| &= 3 \\ |\{\mathbb{N}, \mathbb{Z}, 3, \{4\}\}| &= 4 \\ |\varnothing| &= 0. \end{aligned}$$

1.2. **Subsets.** Note that every element in $\mathbb{N}$ is an element of $\mathbb{Z}$. When this happens, we write $\subseteq$, and we say one set is a *subset* of the other.

Definition 1.7. Given two sets A and B, we write $A \subseteq B$ if $x \in A$ implies that $x \in B$. In words, every element in A is also an element in B. We write $A \subsetneq B$ if A is *not* a subset of B.

Example 1.8. We have that $\mathbb{N} \subseteq \mathbb{Z}$.

Question 1.9. Given two sets A and B, how would you argue that A is *not* a subset of B?

You just have to find some element in A that is not in B.

Example 1.10. To argue that $A = \{3, 6, 8, 1\}$ is not a subset of $B = \{2, 6, 8, 1, 5\}$, we see that $3 \in A$ but $3 \notin B$. Therefore $A \subsetneq B$.

Example 1.11. Let $A = \{1, 2, 3\}$. Is it true that $\varnothing \subseteq A$?

Yes! The condition that $\varnothing \subseteq A$ means that for every $x \in \varnothing$ we have that $x \in A$. Since $\varnothing$ has no elements, this is true.[4] In fact $\varnothing \subseteq S$ for *any* set S.

[3] People who know a little CS, we might think about this as an infinite for loop (for all $n \in \mathbb{N}$, add $3 \cdot n$ to the set we're building, and let A be the resulting output). Obviously this wouldn't terminate on a computer, but we're mathematicians so we can let things happen infinitely many times and keep moving!

[4] We refer to statements like this as *vacuously true* – they're true because no elements exist to check the conditions on. For example I might say "every number which is both even and odd is equal to 7." This is a true statement, not because 7 is both even and odd, but because no numbers are both even and odd.

1.3. **Set equality.**

Question 1.12. What does it mean for two sets to be equal?

Example 1.13. We claim that $\{4, 1, 0\} = \{0, 1, 4\}$.

Answer 1.14. Two sets A and B are equal if they have the same elements. Phrased differently, $x \in A$ implies $x \in B$ and $x \in B$ implies $x \in A$. That is, $A \subseteq B$ and $B \subseteq A$.

1.4. **Operations with sets.** Given two sets A and B we denote by $A \cup B$ their *union*, meaning the set of all elements in A or in B.

$$A \cup B = \{x \colon x \in A \text{ or } x \in B\}.$$

Example 1.15. We have that

$$\{1, 2, 3\} \cup \{4, 5, 6\} = \{1, 2, 3, 4, 5, 6\}.$$

Note we don't allow repeats, so

$$\{1, 2, 3\} \cup \{3, 4\} = \{1, 2, 3, 4\}.$$

Given two sets A and B, we denote by $A \cap B$ their *intersection*, meaning the set of all elements in both A **and** B:

$$A \cap B = \{x \colon x \in A \text{ and } x \in B\}.$$

Example 1.16. We have

$$\{1, 2, 3, 4\} \cap \{3, 4, 5, 6\} = \{3, 4\}.$$

Question 1.17. What is

$$\{1, 2, 3\} \cap \{4, 5, 6\}?$$

It is the empty set! There are no elements in both sets.

Finally we denote by $A - B$ their *difference*, meaning

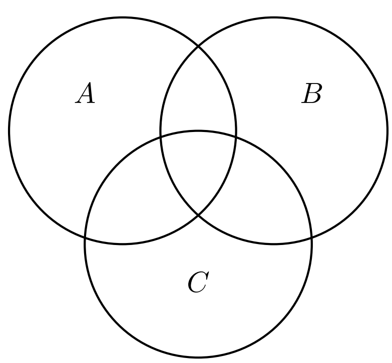
$$A - B = \{x \colon x \in A \text{ and } x \notin B\}.$$

For instance

$$\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}.$$

Note that difference depends on the order of sets! We always have that $A \cup B = B \cup A$ and $A \cap B = B \cap A$, but $A - B$ and $B - A$ might be different sets.

Venn diagrams are a great way to visualize sets and their overlaps:



1.5. **Power sets.** Given a set A, we denote by
$$\mathcal{P}(A) := \{X \colon X \subseteq A\}$$
the *power set* of A, meaning the set of all subsets of A.

Question 1.18. What is the power set of $\{1, 2\}$?

It is the set
$$\mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}.$$
Don't forget that $\varnothing \subseteq S$ and $S \subseteq S$ for every set S.

Question 1.19. If S has cardinality n, what do you think the cardinality of the power set $\mathcal{P}(S)$ is? Think about this.

1.6. **The real numbers.** We denote by $\mathbb{R}$ the set of *real numbers*. These are numbers we think about as lying on the number line, but need not be rational. For instance $\pi \in \mathbb{R}$ but $\pi \notin \mathbb{Q}$.[5] It's not super easy to define $\mathbb{R}$ formally, so we'll come back to this later in the class.

We define *intervals* to be subsets of $\mathbb{R}$. You may have seen the notation $[0, 1]$ before. This refers to the *closed interval* between zero and one. Explicitly in terms of set builder notation, we would write:
$$[0, 1] = \{x \in \mathbb{R} \colon 0 \leq x \text{ and } x \leq 1\}.$$
We also have open intervals, denoted by (a, b). For instance
$$(2, 3) := \{x \in \mathbb{R} \colon 2 < x \text{ and } x < 3\}.$$

1.7. **Cartesian products.**

Definition 1.20. An *ordered pair* is a tuple of two things (x, y).

Definition 1.21. Given two sets A and B, we define their *(Cartesian) product* denoted $A \times B$ by
$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Example 1.22. If $A = \{x, y, z\}$ and $B = \{1, 2\}$ then
$$A \times B = \{(x, 1), (x, 2), (y, 1), (y, 2), (z, 1), (z, 2)\}.$$

Example 1.23. When we graph things on the xy-plane, we are thinking about a *subset* of $\mathbb{R}^2$
$$\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}.$$
In particular if $y = f(x)$ is a function, we could graph the subset
$$\left\{(x, y) \in \mathbb{R}^2 : y = f(x)\right\}.$$
This is most of what we do in high school algebra - studying subsets of $\mathbb{R}^2$ of this form.

Observe: For any two finite sets A and B, we have that
$$|A \times B| = |A| \cdot |B|.$$

We can iterate multiplying sets, for instance if $A_1, A_2, \ldots, A_n$ are all sets, then we denote by
$$A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) : x_i \in A_i \text{ for each } i = 1, \ldots, n\}.$$
We might use shorthand notation for this:
$$\prod_{i=1}^{n} A_i = A_1 \times \cdots \times A_n.$$

[5] This is not super easy to prove, but we'll see examples later of irrational numbers.

This type of notation is common also for unions and intersections of more than two sets:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup \cdots \cup A_n$$
$$\bigcap_{i=1}^{n} A_i = A_1 \cap \cdots \cap A_n.$$

1.8. **Index sets.** If we have sets $A_1, A_2, \ldots, A_n$, another way to phrase this is that we have sets *indexed* over the set $I = \{1, 2, \ldots, n\}$. In other words for each $i \in I$ we have a set A_i. In that way we can rewrite the operations above as

$$\prod_{i \in I} A_i, \qquad \bigcup_{i \in I} A_i, \qquad \bigcap_{i \in I} A_i.$$

From this perspective it's not really important that I was a set of natural numbers. We can have sets A_i indexed over *any* index set I.

1.9. **Complements.** If $B \subseteq A$ is a subset, we denote by B^c the *complement* of B in A, meaning everything that is in A and not in B:

$$B^c = \{x \in A : x \notin B\}.$$

Note that Hammack writes this as $\bar{B}$.

2. Axiomatic rules for sets

We've mentioned that it's hard to define sets, but that they satisfy certain rules. We'll lay these out now. These rules were developed by Zermelo and Fraenkel in the first few decades of the 20th century, building on work in formal logic and set theory in the 19th century. We call these axioms **ZF** after Zermelo and Fraenkel, and there are 8 axioms in total.

Definition 2.1. A set X is a *pure set* or a *hereditary set* if all of its elements are themselves sets, and all of the elements of those sets are sets, and so on.

Example 2.2. The empty set $\varnothing$ is vacuously a pure set. The set $\{\varnothing\}$ or $\{\varnothing, \{\varnothing\}\}$, are also pure, for instance.

Pure set theory: Let's treat this like a game, and temporarily forget everything we're allowed to do with sets. Our pieces are pure sets, and here are the rules.

(1) given any two sets A and B, you are allowed to ask if they are equal, and the answer is either true or false.[6]
(2) given any two sets A and B you're allowed to ask if $A \in B$, and the answer is either true or false.
(3) you're allowed to use as many variables as you want to represent sets
(4) you're allowed to negate any statement and ask if it is true or false (i.e. is it true that $A \neq B$)
(5) you can make "for all" and "implies"statements, like "for all $X \in A$" this "implies" that $X \in B$ (meaning $A \subseteq B$)
(6) you can make "there exists" statements like "there exists $x \in A$ so that $X \notin B$" (meaning $A \not\subseteq B$).

[6]Just like anything in math, we could ask what happens if we remove some of the basic building blocks. What happens if we let statements like $A \in B$ admit another truth value - not true or false but something else? What if, for instance, the *truth* of a statement is a number in the interval $[0, 1]$ where 0 is absolutely false and 1 is absolutely true, but we can have intermediate stages? These kinds of questions lead us to something called *fuzzy logic*, a fascinating detour we sadly won't explore in this class.

On top of these ground rules we're going to have some *axioms*. An *axiom* is like a mathematical rule. They are some base facts that you take for granted, and build mathematics off of. By no means are the axioms we're laying out here the only axioms you could build mathematics off of, and we're not even necessarily saying they're "true." They just end up leading to a convenient formulation of a lot of things we want to do in math.

Note 2.3. The numbering here is not a standard thing, I'm just using it to keep track of stuff easier.

ZF1: *(Axiom of extensionality)* Two sets are equal if they have the same elements.

ZF2: *(Axiom of union)* Unions of sets exist.[7]

ZF3: *(Axiom of power set)* Power sets exist – if A is a set then $\mathcal{P}(A)$ is a valid set.

ZF4: *(Axiom of pairing)* If A and B are sets, then the set $\{A, B\}$ exists.

Corollary 2.4. If A is a set then $\{A\}$ is a set.

Proof. Since A is a set, we can apply the axiom of pairing to A and itself to form the set $\{A, A\}$. Since sets can't have repeated elements, this set $\{A, A\}$ guaranteed by the axiom of pairing only has *one* element, so we abbreviate it $\{A\}$. □

ZF5: *(Axiom of regularity)* If S is a nonempty set, then it contains an element $T \in S$ so that T and S are disjoint sets (have no elements in common).

This is maybe nonintuitive but it has some important applications.

Corollary 2.5. No set can contain itself as an element.

Proof. Let A be any set, and consider the set $S = \{A\}$. By **ZF5**, S contains an element that is disjoint from itself, and since S only has one element, this implies that S is disjoint from A. In other words A and $\{A\}$ have no elements in common, so in particular $A \notin A$. □

ZF6: *(Axiom schema of specification)* You can build sets with set builder notation.[8]

Explicitly, **ZF6** says that the following type of set building is allowed:[9]

$$\{x \in A : \text{something about } x \text{ is true}\}.$$

But this type of set building is not allowed:

$$\{x : \text{something about } x \text{ is true}\}.$$

Why can't we let the latter exist?

Russell's paradox: Suppose we're allowed to build sets of that form, and we take

$$S = \{x : x \notin x\}.$$

We've already seen that no set can contain itself, so $x \notin x$ for every set x. In particular S contains *every set*. But S itself is a set, which means $S \in S$. But also $S \notin S$. These can't both be true, so we've broken math!

[7] The precise statement is if A is a pure set, there exists a set $\cup_{B \in A} B$ which is a union of all the elements of A (the most precise statements says there is a set *containing* $\cup_{B \in A} B$, and we can shorten this to $\cup_{B \in A} B$ using the axiom of pairing). For CS people, this is an axiomatization of the process of *flattening* a set or a list.

[8] We're being vague here – **ZF6** tells you more concretely *what kinds of formulas* you're allowed to use in set builder notation, but let's treat this as a black box for the time being.

[9] We're being intentionally vague with this "something about x." The precise things that are allowed to be here are what are called *first order formulas*. We'll get into these more next week.

It's generally advisable not to break math, so we exclude sets built like this. The point is not whether $S \in S$ or whether $S \notin S$, the point is such a set S *cannot be allowed to exist* if we want a logically consistent framework of math.

Barber's paradox (a common application of Russell's paradox): A barber cuts everyone's hair who doesn't cut their own hair. Does the barber cut their own hair?

ZF7: *(Axiom schema of replacement)* The domains of functions are sets (roughly speaking).

ZF8: *(Axiom of infinity)* There exists a set with infinitely many elements.

There is a 9th mysterious axiom, called the *axiom of choice*. This isn't one of the ZF axioms, so when we use it we often refer to **ZFC** which is ZF + Choice. We won't go into this as much in this class, but it will become super important later in proof-based mathematics.

3. Logic

A *statement* is any mathematical sentence that can definitively evaluated as true or false.

Here are some examples of statements:

(1) It is Monday today
(2) The number 2 is even
(3) The number 2 is not even
(4) There exists a finite subset of X.
(5) Every natural number is divisible by a prime number
(6) Every subset of an infinite set is infinite.

We can evaluate each of these as true or false.

Let P be a mathematical statement. Then we can assign it a *truth value* meaning an element of the set $\{T, F\}$ where T stands for true and F stands for false.

We can *negate* mathematical statements, which swaps the truth value of the statement. We denote this new statement by $\neg P$ (Hammack writes $\sim P$)

P	$\neg P$
It is Monday today	It is not Monday today
The number 2 is even	The number 2 is not even
The number 2 is not even	The number 2 is even

Pause – what happened here? Let P be "the number 2 is even." Then we just said

$$\neg\neg P \text{ is the same statement as } P.$$

This is called *double negation elimination*.[10] It's an admissible rule in our logical framework that we can cancel two negation symbols when they appear right next to each other.

Let's keep negating:

P	$\neg P$
There exists a finite subset of X	There does not exist a finite subset of X
or	For every subset of X, it is not finite.

Interesting – when we negate a "there exists" statement, we get a "for every" statement.

Let's keep negating:

[10]There exist frameworks of logic that *explicitly reject this*, but classical logic accepts it and so will we in this class.

P	$\neg P$
Every natural number is divisible by a prime number	Not every natural number is divisible by a prime number
or	There exists a natural number which is not divisible by a
or even clearer:	There exists a natural number which is not divisible by

Same deal – negating an "every" statement gets us a "there exists" statement. Finally:

P	$\neg P$
Every subset of an infinite set is infinite	Not every subset of an infinite set is infinite.