LINES ON REAL QUINTIC THREEFOLDS

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Abstract. It is well-known that a generic complex quintic threefold has exactly 2875 lines. For a real quintic threefold, work of Finashin-Kharlamov and Okonek-Teleman shows that at least 15 of the 2875 lines must be defined over the reals. We answer the question, floated explicitly by Sottile some 25 years ago, of how many of the 2875 lines can be real.

1. Introduction

In 1849, Salmon and Cayley proved that there are 27 lines on a smooth complex cubic surface [Cay49], and not a decade later Schläfli had established the possible counts over the reals: a real smooth cubic surface can have 3, 7, 15 or 27 real lines [Sch58]. Schubert established the count of 2875 lines on a generic quintic threefold [Sch86, p. 72] - this was made rigorous in the 20th century (see e.g. [Har79, Kat86]). Nevertheless the analogous question of reality has not been resolved. The problem of how many of the 2875 lines can be real was floated explicitly as an open question by Sottile in [Sot03, §3.1]. To our knowledge the only progress towards this result is work of Finashin and Kharlamov and parallel work of Okonek and Teleman which yields a lower bound of 15 real lines on a real quintic threefold [FK13, OT14]. While neither set of authors explore whether this bound is sharp, we will see that it is.

Unlike lines on cubic surfaces, the Galois group of lines on quintic threefolds is full symmetric [Har79]. Thus we may expect the lower bound coming from signed real enumerative geometry to be the only constraint (c.f. the discussion in [FK13, §5.1]), and indeed this is true. Our main result is:

Theorem 1.1. Let d be any odd number with $15 \leq d \leq 2875$. Then there exists a real quintic threefold with exactly d real lines and $2875 - d$ complex lines.

The proof strategy is straightforward: by analyzing the space of generic real quintic threefolds, we can prove a wall-crossing argument that shows every odd number between a minimum and maximum value is attained (Lemma 2.3). We then numerically certify the lower bound of 15 is attained, and the upper bound is 2875 is attained as well.

1.1. **On certification.** Certification can be done a few ways, using HomotopyContinuation's `certify` method (you have to check on multiple charts), or with `msolve` (if you have some time to kill). Attached to this project is an AI-assembled repo with a short `python-flint` installation. Working with exact dyadic rational coefficients for our quintic threefolds we can certify the provided lines are distinct, and either real (by the Krawczyk test in Arb) or complex, as needed:

`https://github.com/tbrazel/real-quintic-threefolds`

1.2. **Acknowledgements.** This paper was written while the author was a postdoc at SLMath. The code used to hunt for the quintic with all real lines was run on SLMath's high-performance cluster for about a month (an estimated 23,000 core-hours). AI was used to dramatically speed up the code written for this search.

2. Geometric setting

Recall that a *quintic threefold* is the vanishing locus of a degree 5 polynomial $F(x_0, x_1, x_2, x_3, x_4)$ in 5 projective variables, yielding a three-dimensional variety $V(F) \subseteq \mathbb{P}^4$. A generic quintic threefold

over the complex numbers contains exactly 2875 lines, however "generic" here does not mean smooth, unlike the setting of lines on cubic surfaces. We will see an explicit example in more detail later: the Fermat quintic threefold is smooth but Albano and Katz established that it has infinitely many lines (Definition 3.1). Our first goal is then to analyze the geometry of the space of those quintic threefolds containing exactly 2875 lines. We work over the complex numbers for most of this section, until specified.

Let $V_5 = \mathbb{C}[x_0, x_1, x_2, x_3, x_4]_{(5)} \cong \mathbb{C}^{126}$ be the space of homogeneous quintic forms in five variables. The parameter space of quintic threefolds is then $\mathbb{P}(V_5) = \mathbb{P}^{125}$. Within this parameter space, we have the following subspace of interest:

Notation 2.1. We let $\mathrm{Q} \subseteq \mathbb{P}^{125}$ denote the moduli space of quintic threefolds X which have exactly 2875 distinct lines.

Our goal for this section is to better understand what Q is. The main result of this section is:

Lemma 2.2. We have that Q is open in $\mathbb{P}^{125}$ and its complement $D := \mathbb{P}^{125} \setminus \mathrm{Q}$ is an irreducible hypersurface. Moreover if $D^\circ \subseteq D$ is the locus of quintic threefolds with exactly 2874 lines, one of which has mutiplicity two and the remaining 2873 are reduced, then $D \setminus D^\circ$ has codimension ≥ 2 in $\mathbb{P}^{125}$.

As a corollary of this, we obtain a wall-crossing lemma which provides us the possible number of real lines on a quintic threefold in $\mathrm{Q}(\mathbb{R})$ in terms of a minimum and maximum.

Lemma 2.3. Let $N_{\min}$ and $N_{\max}$ be the minimum and maximum number of real lines on quintic threefolds in Q. Then for any odd number d with $N_{\min} \leq d \leq N_{\max}$, there exists a quintic threefold $X \in \mathrm{Q}(\mathbb{R})$ with exactly d real lines.

Proof. Let $X_{\min}$ and $X_{\max}$ be choices of quintic threefolds in $\mathrm{Q}(\mathbb{R})$ attaining the minimum and maximum number of lines, and let $\gamma\colon [0,1] \to \mathbb{R}\mathrm{P}^{125}$ be a continuous map starting at $X_{\min}$ and ending at $X_{\max}$. We can homotope γ so that it misses $D \setminus D^\circ$ and crosses D° transversely at each point. Therefore the number of real lines changes by ± 2 each time γ crosses D. Thus every odd value between $N_{\min}$ and $N_{\max}$ is attained along the path. □

The remainder of this section is devoted to proving Lemma 2.2. In Section 3 we computationally certify that $N_{\min} = 15$ and in Section 4 we certify that $N_{\max} = 2875$, proving Theorem 1.1.

2.1. **The incidence variety of lines on quintic threefolds.** A line $\ell \subseteq \mathbb{P}^4$ can be considered as a point on the Grassmannian $\mathrm{Gr}(2,5)$. Letting $\mathcal{S}$ denote the tautological rank two subbundle, we have a natural short exact sequence of vector bundles over the Grassmannian:

$$0 \to K \to V_5 \otimes \mathcal{O}_{\mathrm{Gr}} \xrightarrow{\mathrm{res}} \mathrm{Sym}^5 \mathcal{S}^\vee \to 0.$$

Here the morphism res takes a form F and a line ℓ and returns the restricted quintic form $F_{|\ell}$, and K is defined to be its kernel. It is clear that res is surjective at every ℓ, so K is a subbundle of rank 120. Projectivizing K, we obtain the incidence variety of lines on quintic threefolds, which we call I as in [Har79]:

$$I := \mathbb{P}(K) = \{(F, \ell) \in \mathbb{P}^{125} \times \mathrm{Gr}(2,5) \mid F_\ell = 0\}.$$

This is a $\mathbb{P}^{119}$-bundle over $\mathrm{Gr}(2,5)$ hence smooth and irreducible of dimension 125 (c.f. [Har79, p. 713]). An equivalent way to describe I is as follows:

Remark 2.4. Let $E = \mathcal{O}_{\mathbb{P}^{125}}(1) \boxtimes \mathrm{Sym}^5 \mathcal{S}^\vee$ on $\mathbb{P}^{125} \times \mathrm{Gr}(2,5)$. This has a canonical section $\sigma(F, \ell) = F_\ell$. Then $I = Z(\sigma)$ as subschemes of $\mathbb{P}^{125} \times \mathrm{Gr}(2,5)$.

Remark 2.5. Let $\pi\colon I \to \mathbb{P}^{125}$ denote projection off of the incidence variety. Then Q is the sublocus of $\mathbb{P}^{125}$ over which π is étale.

In other words, if R_π denotes the ramification locus of π, then $\mathrm{Q} = \mathbb{P}^{125} \setminus \pi(R_\pi)$. We therefore want to better understand where π ramifies. As R_π is the vanishing of a section of a line bundle (namely $d\pi$), and this section is not identically zero but also admits a vanishing locus, it is clear that $R_\pi \subseteq I$ is a divisor.

Proposition 2.6. There is a short exact sequence of vector bundles on I, given by

$$0 \to TI \xrightarrow{d\pi \oplus d\pi'} T\mathbb{P}^{125} \oplus T\mathrm{Gr}(2,5) \xrightarrow{d\sigma} E_{|I} \to 0. \tag{2.1}$$

Proof. Since restriction along a line is surjective, we have that $d\sigma$ is surjective along I. □

Observe that $d\sigma$ has two components, as it is a map out of a coproduct. We write β for the component at the tangent bundle

$$\beta \colon T\mathrm{Gr}(2,5) \to E_{|I}.$$

This is a square map, i.e. it is between rank six vector bundles over I. A cute abelian categories trick lets us describe R_π scheme-theoretically in terms of β:

Proposition 2.7. Scheme-theoretically, we have that R_π is equal to degeneracy locus of β.

Proof. By Equation (2.1) and the snake lemma, we have that $\mathrm{coker}(d\pi) \cong \mathrm{coker}(\beta)$. Since both $d\pi$ and β are square, their degeneracy loci are cut out as the zeroth Fitting ideal of their cokernels, which are isomorphic. Thus the result follows by [Aut, 07Z8]. □

As an immediate corollary, we can get the divisor class of R_π if we like:

Corollary 2.8. We have that

$$[R_\pi] = 10\square + 6h,$$

where $\square$ is the pullback of the codimension one Schubert variety on $\mathrm{Gr}(2,5)$, and h is the hyperplane class in $\mathbb{P}^{125}$.

Proof. This follows by the Porteous-Thom formula:

$$[R_\pi] = c_1(E) - c_1(T\mathrm{Gr}(2,5)) = (15\square + 6h) - 5\square = 10\square + 6h.$$

□

We give the following local description of β:

Lemma 2.9 (A coordinate description of β)**.** Fix a quintic form F. Then β restricted to $\pi^{-1}(F)$ is the intrinsic derivative of restriction to F. In particular in coordinates with $\ell = \{x_2 = x_3 = x_4 = 0\}$, we can write

$$F = \sum_{i=2}^{4} x_i h_i(x_0, x_1) + O((x_2, x_3, x_4)^2).$$

Let $v \in T_\ell \mathrm{Gr}(2,5)$ be a tangent vector, viewed as a 2×3 matrix of linear forms

$$\begin{aligned} v_2 &= v_{20}x_0 + v_{21}x_1 \\ v_3 &= v_{30}x_0 + v_{31}x_1 \\ v_4 &= v_{40}x_0 + v_{41}x_1. \end{aligned}$$

Then

$$\beta_{F,\ell}(v) = \sum_{i=2}^{4} v_i h_i.$$

We now proceed to argue that R_π only has one component, i.e. it is irreducible. We do this by exhibiting a proper map $R' \to R_\pi$ from some variety R' we will define, which is easier to analyze. We call R' the *space of ramification directions*.

2.2. **The space of ramification directions.** We want to build a space, call it R', which captures not just the ramification of π but also some tangential information about how it ramifies, and we will use this to better understand R'.

We have seen that R_π is given by those points where $\beta_{(F,\ell)}$ has nonzero kernel. Letting $q\colon \mathbb{P}T\mathrm{Gr}(2,5) \to \mathrm{Gr}(2,5)$ be the projection map, we have a bundle map

$$\begin{aligned}\Psi\colon q^*K \otimes \mathcal{O}_{\mathbb{P}T\mathrm{Gr}}(-1) &\to q^*\mathrm{Sym}^5\mathcal{S}^\vee\\ F \otimes v &\mapsto \beta_{(F,\ell)}(v).\end{aligned}$$

If $(\ell,[v]) \in \mathbb{P}T_\ell\mathrm{Gr}(2,5)$ then we can write

$$\begin{aligned}\Psi_v\colon K_\ell &\to \mathrm{Sym}^5\mathcal{S}_\ell^\vee\\ F &\mapsto \beta_{(F,\ell)}(v).\end{aligned}$$

This is well-defined (it doesn't depend on scaling v) so its rank and kernel are well-defined - this observation will be important soon.

We can let $M = \mathbb{P}(q^*K)$, which is alternatively described as a pullback:

$$\begin{array}{ccc} M & \longrightarrow & I \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow \\ \mathbb{P}(T\mathrm{Gr}(2,5)) & \xrightarrow[q]{} & \mathrm{Gr}(2,5). \end{array}$$

As a $\mathbb{P}^5$-bundle over I, it is clear that M is smooth, projective, and irreducible of dimension 130.

Let's look at M a bit closer. A point in M is a triple $([F],\ell,[v])$, where $F \in V_5$, ℓ is a line contained on $V(F)$, and $v \in T_\ell\mathrm{Gr}(2,5)$ is a tangent direction to the line. The variety M carries two tautological line bundles:

(1) $\mathcal{O}_M(-1)$: the tautological line bundle whose fiber is F, it is pulled back from $\mathcal{O}_{\mathbb{P}^{125}}(-1)$
(2) $p^*\mathcal{O}_{\mathbb{P}T\mathrm{Gr}}(-1)$, whose fiber is v.

By pulling Ψ back to M along p, it induces a section of the rank six bundle

$$\mathcal{O}_M(1) \otimes p^*\mathcal{O}_{\mathbb{P}T\mathrm{Gr}}(1) \otimes p^*q^*\mathrm{Sym}^5\mathcal{S}^\vee.$$

Let R' be the zero scheme of this section. As a set,

$$R' = \{([F],\ell,[v]) \in M \mid \beta_{(F,\ell)}(v) = 0\}.$$

As a scheme it is locally cut out by six equations in M. We name the projections off of R', call them q_1 and q_2:

$$\begin{aligned}q_1\colon R' &\to I\\ q_2\colon R' &\to \mathbb{P}T\mathrm{Gr}(2,5).\end{aligned}$$

Proposition 2.10. We have that $q_1(R') = R_\pi$ as sets.

Proof. A point $([F],\ell,[v])$ lies in R' exactly when $\beta_{(F,\ell)}(v) = 0$, and since $v \neq 0$ this is equivalent to saying that $([F],\ell) \in R_\pi$ by Proposition 2.7. □

To study R' we look at the other projection q_2 landing in $\mathbb{P}T\mathrm{Gr}(2,5)$. We first remark that we can stratify this target space.

Notation 2.11. A point $v \in T_\ell\mathrm{Gr}(2,5)$ can be viewed as a 2×3 matrix under the isomorphism $T_\ell\mathrm{Gr}(2,5) \cong \mathrm{Hom}(\mathcal{S}_\ell, \mathbb{C}^5/\mathcal{S}_\ell)$. Thus v has a well-defined *rank*. This stratifies the total space of the projective tangent bundle as

$$\mathbb{P}T\mathrm{Gr}(2,5) = P_2 \amalg P_1,$$

where P_r is the space of rank r tangent directions.

Proposition 2.12. We have that $P_1 \subseteq \mathbb{P}T\mathrm{Gr}(2,5)$ is closed, and is a $\mathbb{P}^1 \times \mathbb{P}^2$-bundles over $\mathrm{Gr}(2,5)$, hence dimension 9. Its complement P_2 is open and irreducible, of dimension 11.

Proof. We have that P_1 is closed since it is cut out by 2×2-minors. The fiber of P_1 over a point in the Grassmannian is the space of 2×3 matrices of rank one up to scaling, which is a Segre variety $\mathbb{P}^1 \times \mathbb{P}^2$. □

Proposition 2.13. We have that $R' = q_2^{-1}(P_1) \cup q_2^{-1}(P_2)$, where:

(1) $q_2^{-1}(P_1)$ has dimension ≤ 123
(2) $q_2^{-1}(P_2)$ is a $\mathbb{P}^{113}$-bundle over P_2, hence nonempty and irreducible of dimension 124

Proof. Working with the local equation for β, we can see that the rank of Ψ_v is 5 if and only if $v \in P_1$, and the rank of Ψ_v is 6 if and only if $v \in P_2$.

The kernel of Ψ over P_1 is a subbundle of rank 115, so $\dim q_2^{-1}(P_1) = 9 + 114 = 123$. Similarly, the kernel of Ψ over P_2 is a subbundle of rank 114, and $q_2^{-1}(P_2)$ is its projectivization, from which the result follows. □

Corollary 2.14. We have that R' is irreducible of dimension 124, and therefore R_π is irreducible.

Proof. Since M is smooth and irreducible of dimension 130, and R' is cut out by 6 equations locally, any irreducible component of R' has dimension at least 124. We can write $R' = q_2^{-1}(P_1) \cup \overline{q_2^{-1}(P_2)}$, and the dimension counts above show us that $R' = \overline{q_2^{-1}(P_2)}$, which is irreducible. Since R_π is the image of R' along a proper map, it is irreducible as well. □

2.3. **The proof of Lemma 2.2.** We recall the following result of Harris:

Proposition 2.15. There exists a quintic threefold F_0 with exactly 2874 lines, one of multiplicity two and the remaining 2873 reduced [Har79, III.2].

This is in fact the generic ramification behavior. More precisely, let $D = \pi(R_\pi)$ be the locus over which π fails to be étale. Let $D^\circ \subseteq D$ be the locus of those quintic threefolds with exactly 2874 lines. Then:

Proposition 2.16. We have that

(1) $D \subseteq \mathbb{P}^{125}$ is an irreducible hypersurface
(2) D is smooth along D°, and $D \setminus D^\circ$ has codimension ≥ 2 in $\mathbb{P}^{125}$.

Proof. For (1) since π is proper and R_π is irreducible, we have that D is closed and irreducible, of dimension ≤ 124. By proper pushforward and projection, we can compute that

$$67000 = \int_I [R_\pi]\pi^* 124 = \int_{\mathbb{P}^{125}} \pi_*[R_\pi] \cdot h^{124}.$$

The right hand vanishes if D had dimension strictly less than 124, which is a contradiction. Thus D is a hypersurface. We have that (2) follows from a local computation and a dimension count. □

Remark 2.17. One can prove that $\pi_{|R_\pi}$ is birational, and R_π is reduced, hence D has degree 67000. We omit the proof as it is not needed for our main results.

3. Realizing the lower bound

In this section we exhibit an explicit quintic threefold $X_{\min} \in \mathrm{Q}(\mathbb{R})$ with 15 real lines. Before we do this, we take a moment to discuss the beautiful history of this lower bound.

3.1. **Establishing the lower bound.** The lower bound of 15 comes from the world of *signed real enumerative geometry*. Roughly speaking, this is a family of techniques which repair the failure of Schubert's conservation of number over $\mathbb{R}$, at the cost of weighting solutions to enumerative problems by a sign. One of the first major results is Segre's observation that lines on real cubic surfaces can be divided into *hyperbolic* and *elliptic* lines, depending on how the tangent plane varies as we traverse the line on the surface. Segre argues the number of real hyperbolic lines minus the number of real elliptic lines is always 3, in particular this establishes a lower bound on the possible number of real lines.

This can be made rigorous by use of a so-called *absolute Euler class*, which is a characteristic class for real vector bundles living in twisted integral cohomology. In two papers released around the same time [FK13, OT14], the authors compute the absolute Euler classes for the real vector bundle of lines on a degree $2d-3$ hypersurface in $\mathbb{P}^d$, showing that it is the falling factorial $(2d-3)!!$. When $d=4$, we obtain a lower bound of 15 real lines on a real quintic threefold.

This program of weighting solutions by signs (or square classes) can be generalized to other fields using motivic homotopy theory. The quadratically enriched count of lines on quintic threefolds is in a lovely paper of Pauli [Pau22].

3.2. **About the Fermat.** We exhibit a real quintic threefold with exactly 15 real lines, and it arises as a perturbation of the Fermat quintic. We take a moment to discuss this very unique and beautiful threefold.

Definition 3.1. The *Fermat quintic threefold* is given by the equation

$$F = x_0^5 + \ldots + x_4^5.$$

Albano and Katz prove that the Fermat contains infinitely many lines [AK22]. In particular they observe that, by restricting to a hyperplane of the form $x_i + \zeta_5 x_j = 0$ for any (not necessarily primitive) 5th root of unity ζ_5, the $x_i^5 + x_j^5$ term vanishes, and we reduce to $x_k^5 + x_\ell^5 + x_m^5$. This contains the cone with vertex $e_i - \zeta_5 e_j$ over the Fermat planar quintic curve, hence infinitely many lines. When $\zeta_5 = 1$ we get 10 real cones. Their proof argues that if ℓ is any line on the Fermat, it necessarily lies on one of these cones. In fact there are 375 lines which lie on exactly *two* of these cones, and exactly 15 are real.

Example 3.2 (An example cone)**.** We can consider the cone $K_{01} = V(F) \cap \{x_0 = x_1 = 0\}$. The lines on this are of the form $\mathrm{span}\{v, c\}$ where $v = [1:-1:0:0:0]$ and $c = [0:0:c_2:c_3:c_4]$ is any point on the plane quintic $C = \{x_2^5 + x_3^5 + x_4^5 = 0\}$ on the plane $\{x_1 = x_2 = 0\}$. We can write a line on this cone as $L = \lambda v + \mu c$ where $[\lambda : \mu] \in \mathbb{C}\mathrm{P}^1$.

The idea for finding a quintic with the fewest number of real lines is to degenerate the Fermat slightly, trying to force as few of the cone lines as possible to be real.

3.3. **The lower bound.**

Theorem 3.3 (Certified)**.** Let $\epsilon = 1/128$, and consider the quintic threefold obtained as a deformation of the Fermat:

$$G_\epsilon = \left(\sum_{i=0}^{4} x_i^5\right) + \epsilon \left(\sum_{i<j} (x_i - x_j)^3 \sum_{k \neq i,j} x_k^2\right).$$

Then $V(G_\epsilon)$ has exactly 15 real lines and 2860 complex lines.

4. Realizing the upper bound

The upper bound took far more work, running on a computer cluster for about a month. We take a brief moment to discuss what was tried before finding a solution obtaining the upper bound.

4.1. **Try #1: naive hill-climbing.** The most naive thing one can think to do is pick 126 random coefficients to get a quintic threefold in $\mathrm{Q}(\mathbb{R}) \subseteq \mathbb{P}^{125}$ then try to see how many of its solutions are real. It's unclear how long it would take to find a quintic attaining the upper bound in this way, but heuristically speaking for any reasonable definition of randomness the chambers consisting of such quintic threefolds will have tiny volume compared to the whole space, so this approach is doomed.

Far better is to pick a starting quintic threefold (call this the *seed*) and numerically solve for its lines. We then try to cross over D by making small nearby jumps in $\mathbb{R}\mathrm{P}^{125}$, only passing into a chamber where the number of lines increases. This is called *hill climbing*, and it is a common technique in numerical analysis. This process can be easily parallelized by letting multiple seeds run at once (we were working with 40 seeds running simultaneously) thus it is well-suited to cluster computing.

The way a jump works is we deform our quintic threefold in some direction by ϵ, then solve for the lines on this new quintic threefold. We don't have to solve fresh - we can leverage homotopy continuation to homotope our previous solutions to this new one. There is a robust software ecosystem now for doing these types of computations [BT18].

Nevertheless at any stage of the search at a given seed, it lives inside a 125-dimensional space, so there are many possible tangent directions to move in. How do we pick one which we expect will drive up the number of real lines?

An answer comes from kinematic robotics! Dietmaier outlined, in his 1998 paper, a way to drive up the number of real solutions to a problem (in his case finding Stewart-Gough platforms with all real assembly modes) while retaining the previous real solutions [Die98].

In our context, Dietmaier's algorithm takes the following shape: if we have a real quintic threefold with less than 2875 real lines, we can pick a complex conjugate pair of lines, then ask for a real change in the coefficients which would make the imaginary part of these lines vanish to first order. We then take an overshot in that direction, track solutions and see if the number of real solutions increased. If that fails we half the direction of the jump and try again. If both of those steps fail, we try again with a different complex conjugate pair of lines. If the process fails for all complex conjugate pairs of lines, we consider ourselves at a local maximum (whether or not we actually are) and then "kick" the solution further away by adding on a randomized Kostlan quintic, then we try again.

This process got us from the small counts we could find locally (57-97 real lines) up to 751 in two days then to 1071 two days after that. This process stalled at 1071 for about a week. This is decent evidence for the heuristic that the chambers with more real lines are tiny - small kicks away from decent quintics would lose hundreds of lines at once.

4.2. **Try #2: degeneration in a symmetric family.** The expectation is that, by preserving symmetry, lines will group together in orbits over the reals or complex numbers. These families are very much not in general position with respect to the walls, so we don't see the generic wall-crossing behavior of ± 2, instead we see more substantial jumps, which is one upshot. The other major upshot is that insisting on symmetry dramatically reduces the search space, so we could anticipate that hill-climbing with Dietmaier's algorithm would run faster.

There are a number of real symmetries we could try to study - Oguiso and Yu have worked out the symmetry groups for smooth complex quintic threefolds [OY19], and although we don't care about smoothness as much here, we can still try to work within some of the families they identify. While A_5 and S_5 families have a 7-dimensional search space, they don't have isolated lines so they are unusable, but other families gave immediate payoffs:

family	dimension of search space	result
C_5	26	no significant progress
D_5	16	$75 \to 865$ in 15 minutes, 2115 overnight
F_{20}	9	$135 \to 1895$ in 15 minutes, 2835 overnight

Here F_{20} is the Frobenius group of order 20, which here is the group $\mathrm{AGL}(1,5)$ of affine maps of $\mathbb{F}_5$ acting on the coordinates of $\mathbb{P}^4$ as follows:

$$F_{20} = \left\langle \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \right\rangle \leq \mathrm{PGL}_5(\mathbb{R}).$$

The F_{20} family was by far the most promising, getting us 40 away from the upper bound. Unfortunately it stayed at 2835 and never got any higher. The lines were laid out as an F_{20}-set as follows:

$$126[F_{20}/e] + [F_{20}/C_4] + 35[F_{20}/C_2].$$

The lines with trivial and C_4 isotropy were all real. Among the 350 lines with C_2 isotropy, they were divided into 5 groups of 70 depending on the 5 different order two elements in F_{20}. Within each of these 70 we could only find 62 which were real. Random hill climbing never increased this count passed 62. It is still unclear whether an F_{20}-symmetric real quintic threefold can have more than 2835 real lines, it would be interesting to know the answer.

4.3. **Try #3: degeneration from planes.** With the F_{20} idea stalled out, we needed a new approach. When Segre studied lines on real cubic surfaces in the 40's, he did it by deforming them to unions of hyperplanes. There is a combinatorial approach to counting curves on hypersurfaces via such degenerations - this is inspired by work of Katz [Kat83] (see also this unpublished paper of Nishinou [Nis15]).

Consider a quintic form $f \in V_5$, and study the quintic threefold

$$X_t = x_0x_1x_2x_3x_4 + tf,$$

where t is a real parameter. As $t \to 0$ the quintic degenerates into a union of hyperplanes.

Let $H_i = \{x_i = 0\}$ be one of these hyperplanes. Inside this 3-dimensional hyperplane, we have 4 faces $H_i \cap H_j$, meeting along six edges $H_i \cap H_j \cap H_k$, giving a tetrahedron. We can ask: which lines $\ell \subseteq H_i$ are limits of lines on X_t?

Picking two points $p, q \in H_i$, we can parametrize a line as

$$\ell = \{up + vq\}_{[u:v]\in\mathbb{P}^1} \subseteq H_i.$$

If ℓ is a limit of a line in X_t, we can think about p and q as functions of t, where $p(t)$ and $q(t)$ span a line on X_t, and $p(0) = p$ and $q(0) = q$. We can approximate ℓ on X_t by its first order differential at t. That is, we can write

$$\ell_t \colon [u,v] \to up(0) + utp'(0) + vq(0) + vtq'(0).$$

Since $p_i = q_i = 0$, the ith coordinate is t times a linear form in u and v:

$$x_i = t(p_i'(0)u + q_i'(0)v) =: t\lambda(u,v).$$

Any other coordinate x_j for $j \neq i$ is given by

$$x_j = up_j + vq_j + t(up_j'(0) + vq_j'(0)).$$

Plugging these into X_t, we get the restricted equation for $X_{t|\ell_t}$, which is

$$t\left(\lambda(u,v)\prod_{j\neq i} x_j + f\right)_{\ell_t} + O(t^2) = 0.$$

Dividing by t and setting it equal to zero we get that

$$\lambda \prod_{j \neq i} x_j + f = 0 \text{ along } \ell.$$

This is an equality of binary quintics along ℓ. We can then solve for λ in terms of the four points of intersection $\ell \cap H_j$. After doing this, we obtain the following:

Proposition 4.1. A line $\ell \subseteq H_i$ is a limit line of X_t if and only if ℓ misses all the edges $H_i \cap H_j \cap H_k$ and ℓ intersects the four plane quintic curves $\{f = 0\} \cap H_i \cap H_j$.

For generic f, each hyperplane H_i gives exactly 575 lines satisfying the conditions of the proposition. Each system is square, and the systems have some overlap. Nevertheless we can conclude that if f is real and all 2875 lines are real nondegenerate solutions to the above, then for sufficiently small t we have that $X_t \in \mathrm{Q}(\mathbb{R})$ and all 2875 of its lines are real.

This motivates the idea to hill climb for f in the problem above. This becomes much simpler (we probably should have done this from the beginning). At the end we obtain a certified solution.

4.4. **The upper bound.**

Theorem 4.2 (Certified)**.** Let F be the quintic threefold

$$F = x_0x_1x_2x_3x_4 + t(L_1L_2L_3L_4L_5 + \epsilon g),$$

where the L_i are linear forms below, g is a quintic, $t = 2^{-47}$ and $\epsilon = 2^{-60}$. The forms are given dyadically as

$$\begin{aligned}
L_1 = &-2669373918683 \cdot 2^{-54}\, x_0 + 2844428917437 \cdot 2^{-51}\, x_1 - 274877683507 \cdot 2^{-38}\, x_2 \\
&- 138982213817 \cdot 2^{-51}\, x_3 + 2369681452667 \cdot 2^{-55}\, x_4 \\
L_2 = &-1333592439 \cdot 2^{-44}\, x_0 + 1589808745043 \cdot 2^{-41}\, x_1 - 12153445906713 \cdot 2^{-44}\, x_2 \\
&+ 8229557088053 \cdot 2^{-51}\, x_3 - 1923136027 \cdot 2^{-38}\, x_4 \\
L_3 = &\ 68179291049 \cdot 2^{-36}\, x_0 - 8805821329235 \cdot 2^{-46}\, x_1 + 13566531708065 \cdot 2^{-56}\, x_2 \\
&- 16801820006429 \cdot 2^{-56}\, x_3 + 232485669647 \cdot 2^{-51}\, x_4 \\
L_4 = &-7326986722063 \cdot 2^{-56}\, x_0 - 208732791821 \cdot 2^{-41}\, x_1 - 6336903 \cdot 2^{-37}\, x_2 \\
&+ 3791747235 \cdot 2^{-32}\, x_3 - 63220798691 \cdot 2^{-37}\, x_4 \\
L_5 = &-1576766541635 \cdot 2^{-41}\, x_0 - 3049284555751 \cdot 2^{-42}\, x_1 + 9414439943 \cdot 2^{-47}\, x_2 \\
&- 8574500166725 \cdot 2^{-47}\, x_3 + 5364839660475 \cdot 2^{-47}\, x_4
\end{aligned}$$

and

$$\begin{aligned}
g = & -6269\cdot 2^{-41}\,x_0^5 + 1208443\cdot 2^{-37}\,x_0^4x_1 - 1879\cdot 2^{-26}\,x_0^4x_2 \\
& - 213606777\cdot 2^{-44}\,x_0^4x_3 - 194274909\cdot 2^{-43}\,x_0^4x_4 + 157941579\cdot 2^{-38}\,x_0^3x_1^2 \\
& + 3692807\cdot 2^{-26}\,x_0^3x_1x_2 + 27151753\cdot 2^{-34}\,x_0^3x_1x_3 + 34804565\cdot 2^{-35}\,x_0^3x_1x_4 \\
& - 8443869\cdot 2^{-27}\,x_0^3x_2^2 - 40171\cdot 2^{-19}\,x_0^3x_2x_3 - 39922743\cdot 2^{-28}\,x_0^3x_2x_4 \\
& - 3878263\cdot 2^{-35}\,x_0^3x_3^2 - 126732795\cdot 2^{-35}\,x_0^3x_3x_4 + 16196451\cdot 2^{-39}\,x_0^3x_4^2 \\
& - 1473967\cdot 2^{-28}\,x_0^2x_1^3 - 228786999\cdot 2^{-32}\,x_0^2x_1^2x_2 - 107398139\cdot 2^{-34}\,x_0^2x_1^2x_3 \\
& - 564547\cdot 2^{-26}\,x_0^2x_1^2x_4 + 193179705\cdot 2^{-31}\,x_0^2x_1x_2^2 - 3460041\cdot 2^{-35}\,x_0^2x_1x_3^2 \\
& + 84393067\cdot 2^{-35}\,x_0^2x_1x_4^2 + 49337\cdot 2^{-21}\,x_0^2x_2^3 + 587\cdot 2^{-12}\,x_0^2x_2^2x_3 \\
& + 1003\cdot 2^{-12}\,x_0^2x_2^2x_4 - 62759\cdot 2^{-27}\,x_0^2x_2x_3^2 - 241465719\cdot 2^{-42}\,x_0^2x_2x_4^2 \\
& - 68799285\cdot 2^{-40}\,x_0^2x_3^3 - 247340901\cdot 2^{-37}\,x_0^2x_3^2x_4 + 261475713\cdot 2^{-37}\,x_0^2x_3x_4^2 \\
& - 171202827\cdot 2^{-42}\,x_0^2x_4^3 - 91605469\cdot 2^{-34}\,x_0x_1^4 + 181971121\cdot 2^{-33}\,x_0x_1^3x_2 \\
& - 162271\cdot 2^{-25}\,x_0x_1^3x_3 - 150784671\cdot 2^{-34}\,x_0x_1^3x_4 + 189723\cdot 2^{-23}\,x_0x_1^2x_2^2 \\
& - 760717\cdot 2^{-30}\,x_0x_1^2x_3^2 + 89813343\cdot 2^{-35}\,x_0x_1^2x_4^2 + 1251609\cdot 2^{-26}\,x_0x_1x_2^3 \\
& - 3864617\cdot 2^{-36}\,x_0x_1x_3^3 - 181146591\cdot 2^{-40}\,x_0x_1x_4^3 + 119\cdot 2^{-26}\,x_0x_2^4 \\
& - 65\cdot 2^{-14}\,x_0x_2^3x_3 - 431\cdot 2^{-16}\,x_0x_2^3x_4 + 9805\cdot 2^{-24}\,x_0x_2^2x_3^2 \\
& + 585\cdot 2^{-20}\,x_0x_2^2x_4^2 + 38537\cdot 2^{-26}\,x_0x_2x_3^3 + 153865681\cdot 2^{-38}\,x_0x_2x_4^3 \\
& - 2468825\cdot 2^{-39}\,x_0x_3^4 - 135036463\cdot 2^{-40}\,x_0x_3^3x_4 + 122247131\cdot 2^{-39}\,x_0x_3^2x_4^2 \\
& - 102492169\cdot 2^{-40}\,x_0x_3x_4^3 + 35011537\cdot 2^{-44}\,x_0x_4^4 + 207440205\cdot 2^{-37}\,x_1^5 \\
& - 81307057\cdot 2^{-27}\,x_1^4x_2 - 837999\cdot 2^{-27}\,x_1^4x_3 + 82912703\cdot 2^{-34}\,x_1^4x_4 \\
& + 77293691\cdot 2^{-27}\,x_1^3x_2^2 - 74427645\cdot 2^{-27}\,x_1^3x_2x_3 - 1\,x_1^3x_2x_4 \\
& - 1516957\cdot 2^{-33}\,x_1^3x_3^2 - 3187025\cdot 2^{-29}\,x_1^3x_3x_4 - 167259763\cdot 2^{-38}\,x_1^3x_4^2 \\
& - 3354021\cdot 2^{-29}\,x_1^2x_2^3 + 77943257\cdot 2^{-27}\,x_1^2x_2^2x_3 + 3913613\cdot 2^{-22}\,x_1^2x_2^2x_4 \\
& - 34334557\cdot 2^{-29}\,x_1^2x_2x_3^2 + 222648719\cdot 2^{-30}\,x_1^2x_2x_4^2 + 18121873\cdot 2^{-39}\,x_1^2x_3^3 \\
& - 2836853\cdot 2^{-32}\,x_1^2x_3^2x_4 + 3111411\cdot 2^{-34}\,x_1^2x_3x_4^2 + 59007209\cdot 2^{-42}\,x_1^2x_4^3 \\
& + 58881\cdot 2^{-34}\,x_1x_2^4 + 3471305\cdot 2^{-27}\,x_1x_2^3x_3 - 15345\cdot 2^{-20}\,x_1x_2^3x_4 \\
& + 268410463\cdot 2^{-32}\,x_1x_2^2x_3^2 - 48947\cdot 2^{-18}\,x_1x_2^2x_4^2 + 246421745\cdot 2^{-36}\,x_1x_2x_3^3 \\
& - 83647061\cdot 2^{-33}\,x_1x_2x_4^3 + 462161\cdot 2^{-39}\,x_1x_3^4 + 53212441\cdot 2^{-40}\,x_1x_3^3x_4 \\
& - 44661605\cdot 2^{-39}\,x_1x_3^2x_4^2 + 132300495\cdot 2^{-42}\,x_1x_3x_4^3 + 3399819\cdot 2^{-43}\,x_1x_4^4 \\
& - 65\cdot 2^{-39}\,x_2^5 - 997843\cdot 2^{-37}\,x_2^4x_3 + 23317\cdot 2^{-33}\,x_2^4x_4 \\
& + 265958087\cdot 2^{-37}\,x_2^3x_3^2 - 361149\cdot 2^{-27}\,x_2^3x_3x_4 + 3318999\cdot 2^{-32}\,x_2^3x_4^2 \\
& - 32697669\cdot 2^{-33}\,x_2^2x_3^3 + 1685439\cdot 2^{-26}\,x_2^2x_3^2x_4 - 224162425\cdot 2^{-33}\,x_2^2x_3x_4^2 \\
& + 2003119\cdot 2^{-28}\,x_2^2x_4^3 + 86699019\cdot 2^{-42}\,x_2x_3^4 + 197501061\cdot 2^{-36}\,x_2x_3^3x_4 \\
& - 3053805\cdot 2^{-30}\,x_2x_3^2x_4^2 + 71093\cdot 2^{-27}\,x_2x_3x_4^3 + 82520539\cdot 2^{-40}\,x_2x_4^4 \\
& + 20183\cdot 2^{-43}\,x_3^5 + 1485523\cdot 2^{-44}\,x_3^4x_4 - 363397\cdot 2^{-41}\,x_3^3x_4^2 \\
& + 182393\cdot 2^{-41}\,x_3^2x_4^3 - 14651\cdot 2^{-44}\,x_3x_4^4 - 1358675\cdot 2^{-48}\,x_4^5
\end{aligned}$$

Then F has exactly 2875 lines, all of them real.

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