A MOTIVIC CRASH COURSE

Abstract. We will give an overview of the construction of motivic spaces and spectra, highlighting the construction of Eilenberg–MacLane spaces and spectra. We will discuss motivic cohomology and the motive attached to a variety. Finally we will present some computations and key properties of motives that will be used later in the seminar.

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0. About

Notes from a talk in the Thursday seminar at Harvard on 10/3/24. The seminar is about the motivic Stenrod algebra and the motivic Wilson space hypothesis.

References: In prepping this talk I drew a lot from Brian Shin's presentation of motivic cohomology from the IWOAT workshop this last summer (2024). Brian learned much of it from Shane Kelly's notes on Voevodsky correspondence [Kel], from Marc Hoyois' paper on the HMH theorem [Hoy15] and the book of Mazza–Voevodsky–Weibel [MVW06]. In prepping this talk I also referenced some fantastic survey papers on motivic cohomology, a classical reference being Deligne's paper on Voevodsky's lectures [Del09], and modern ones including Peter Haine's note [Hai] and Elden Elmanto's IHES minicourse from 2023 [Elm]. The original reference for motives of Eilenberg–MacLane spaces being pure proper Tate (the ultimate goal of this talk) is the original paper of Voevodsky [Voe10]. Other references are cited throughout.

1. Motivic homotopy theory

Assumptions: Here k will be a field, usually perfect. By Sm_k we mean the category of finite type smooth k-schemes.

Let $\mathrm{PSh}(\mathrm{Sm}_k)$ denote the ∞-category of ∞-presheaves on Sm_k. We can think about this as presheaves of "spaces up to weak equivalence."

Definition 1.1. We say $\{U_i \to X\}$ is a *Nisnevich cover* if each $U_i \to X$ is étale, and for every $x \in X$ there exists an i and a $y \in U_i$ mapping to x and inducing an isomorphism on residue fields.

We say $F \in \mathrm{PSh}(\mathrm{Sm}_k)$ is a *Nisnevich sheaf* if, for every Nisnevich cover $\{U_i \to X\}$, the induced map

$$F(X) \to \lim\left(\prod F(U_i) \rightrightarrows \prod F(U_{ij}) \rightrightarrows \cdots\right)$$

is an equivalence.

Denote by $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) \subseteq \mathrm{PSh}(\mathrm{Sm}_k)$ the full subcategory of Nisnevich sheaves.

Definition 1.2. We say $F \in \mathrm{PSh}(\mathrm{Sm}_k)$ is $\mathbb{A}^1$-*invariant* if, for every $X \in \mathrm{Sm}_k$, the projection $X \times \mathbb{A}^1 \to X$ induces an equivalence

$$F(X) \to F(X \times \mathbb{A}^1).$$

We denote by $\mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k) \subseteq \mathrm{PSh}(\mathrm{Sm}_k)$ the full subcategory of $\mathbb{A}^1$-invariant presheaves.

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Definition 1.3. We define the category of *motivic spaces* to be

$$\mathrm{Spc}(k) := \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) \cap \mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k) \subseteq \mathrm{PSh}(\mathrm{Sm}_k).$$

Each of these is an accessible subcategory of $\mathrm{PSh}(\mathrm{Sm}_k)$, and hence the inclusions admit left adjoints

$$L_{\mathrm{Nis}} \colon \mathrm{PSh}(\mathrm{Sm}_k) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k)$$
$$L_{\mathbb{A}^1} \colon \mathrm{PSh}(\mathrm{Sm}_k) \to \mathrm{PSh}_{\mathbb{A}^1}(\mathrm{Sm}_k).$$

We can define the *algebraic n-simplex* over our base field k as

$$\Delta^n := \operatorname{Spec} k[x_0, \ldots, x_1]/(\sum x_i - 1).$$

These form a cosimplicial variety $\Delta^\bullet$.

Proposition 1.4. We can identify $L_{\mathbb{A}^1}$ with the singular chains construction

$$\mathrm{Sing}(F)(X) := \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} F(X \times \Delta^n).$$

Since Δ^{op} is sifted, colimits over it commute with products, hence we see that $L_{\mathbb{A}^1}$ preserves finite products.

To obtain a motivic space from a presheaf it is not enough to $\mathbb{A}^1$ localize and sheafify, since $\mathbb{A}^1$-localization might break the sheaf condition, and sheafifying may violate $\mathbb{A}^1$-invariance. To that end we define *motivic localization* as

$$L_{\mathrm{mot}} := \mathrm{colim}_{n\to\infty} \left(L_{\mathrm{Nis}} \circ L_{\mathbb{A}^1}\right)^{\circ n}.$$

This is adjoint to the inclusion of motivic spaces in presheaves, and preserves finite products.

Example 1.5.

(1) Any $X \in \mathrm{Sm}_k$ gives rise to a motivic space $L_{\mathrm{mot}} h_X$ given by the motivic localization of the representable presheaf. We call this X as well, by abuse of notation.
(2) Any $S \in \mathcal{S}$ gives rise to a constant presheaf, whose motivic localization we also call S.
(3) Any simplicial variety, any stack, etc. give rise to motivic spaces.

There is a pointed version $\mathrm{Spc}(S)_*$, analogous to how we have topological spaces and pointed topological spaces. In this setting we have a smash product, denoted $\wedge$.

There are two kinds of motivic spheres, coming from algebra (things like $\mathbb{G}_m$ and $\mathbb{P}^1$) and things from topology (constant presheaves at simplicial spheres). This is the source of the bigrading in motivic homotopy theory.

Proposition 1.6. We have that $\mathbb{P}^1 \simeq S^1 \wedge \mathbb{G}_m$.

Proof. This comes from the cover

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathbb{P}^1. \end{tikzcd}$$

□

Convention: We write

$$S^{1,0} := S^1$$
$$S^{1,1} := \mathbb{G}_m$$
$$S^{m+n,n} = \mathbb{G}_m^{\wedge n} \wedge (S^1)^{\wedge m}.$$

This gives rise to suspension loops adjunction. Letting $\Sigma^{p,q} := S^{p,q} \wedge -$, we get

$$\Sigma^{p,q} \colon \mathrm{Spc}(k)_* \rightleftarrows \mathrm{Spc}(k)_* : \Omega^{p,q}.$$

This only makes sense when $S^{p,q}$ is a motivic space, i.e. in the range $p \geq q \geq 0$.

Notation 1.7. If we wanted to be pedantic, we would write $\Sigma^{1,0}$ for the suspension $\Sigma = S^1 \wedge -$ above. Often people write $\Sigma^n := \Sigma^{n,0}$, and decorate with a subscript when smashing by a different sphere:

$$\Sigma_{\mathbb{G}_m} := \Sigma^{1,1}$$
$$\Sigma_{\mathbb{P}^1} := \Sigma^{2,1}.$$

1.1. **Stable motivic homotopy.** By inverting smashing with respect to the projective line, we obtain a new category, called the *stable motivic homotopy category*

$$\mathrm{SH}(k) := \mathrm{Spc}(k)_* \left[(\mathbb{P}^1)^{-1}\right].$$

Remark 1.8. This is *not* sheaves of spectra, that is a different category, denoted $\mathrm{SH}^{S^1}(k)$.

This comes with an adjunction

$$\Sigma^\infty \colon \mathrm{Spc}(k)_* \rightleftarrows \mathrm{SH}(k) : \Omega^\infty.$$

The smash product $\wedge$ in $\mathrm{Spc}(k)_*$ gives rise to a symmetric monoidal structure $\otimes$ on $\mathrm{SH}(k)$. Its unit is $\mathbb{1} \in \mathrm{SH}(k)$, the *motivic sphere spectrum.*

If $E \in \mathrm{SH}(k)$ is a motivic spectrum, it gives rise to a cohomology theory — for any motivic space X, we denote by

$$E^{a,b}(X) = \left[X, S^{a,b} \otimes E\right].$$

Example 1.9. We have that algebraic K-theory forms a motivic spectrum, called KGL.

Example 1.10. We will construct a motivic cohomology spectrum HR for each ring R.

1.2. **Thom spectra.**

Definition 1.11. If $\xi \colon V \to X$ is an algebraic vector bundle with zero section $z \colon X \to V$, we denote by

$$\mathrm{Th}_X(\xi) := \frac{V}{V - z(X)}$$

the *Thom space* attached to the bundle. It is an invertible object in $\mathrm{SH}(X)$.

This assignment

$$\mathrm{Vect}(X) \to \mathrm{Pic}(\mathrm{SH}(X))$$
$$\xi \mapsto \mathrm{Th}_X(\xi)$$

is functorial and maps to a group so it factors through its group completion

$$K(X) \to \mathrm{Pic}(\mathrm{SH}(X)).$$

Definition 1.12. We define

$$\mathrm{MGL}_S := \mathrm{colim}_{(X,\xi)} \mathrm{Th}_X(\xi),$$

where the colimit is over all $X \in \mathrm{Sm}_S$ and all $\xi \in K(X)$ so that $\mathrm{rank}(\xi) = 0$.

This is an E_∞-ring, and it is the *universal oriented* ring spectrum.

Remark 1.13. If $k \subseteq \mathbb{C}$ is a subfield of the complex numbers, we have a Betti realization functor

$$\mathrm{Re}_{\mathbb{C}} \colon \mathrm{Sm}_k \to \mathrm{Top},$$

which extends to a functor

$$\mathrm{Re}_{\mathbb{C}} \colon \mathrm{SH}(k) \to \mathrm{Sp}.$$

We have that $\mathrm{Re}_{\mathbb{C}}\mathrm{MGL} = \mathrm{MU}$.

Wilson's theorem dealt with understanding the cohomology of $\Omega^\infty\Sigma^{2n}\mathrm{MU}$. By analogy, we are interested in understanding the cohomology of $\Omega^\infty\Sigma^{2n,n}\mathrm{MGL}$. First we have to say what we mean by cohomology in this context.

2. Motivic cohomology

Goal: For every commutative ring R, define an object $HR^{\mathrm{mot}} \in \mathrm{SH}(k)$ representing *motivic cohomology.*

We want this cohomology theory to be R-linear, to of course admit pullbacks on cohomology from contravariance, but also to have pushforwards along finite maps.

Definition 2.1. Given $X, Y \in \mathrm{Sm}_k$, an *elementary finite correspondence* from X to Y is a span of the form

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X & & Y, \end{tikzcd}$$

where Z is integral, the map $Z \hookrightarrow X \times Y$ is a closed immersion, and $Z \to X$ is finite and dominant over a component of X.

Definition 2.2. Given a ring R, we denote by $\mathrm{Corr}_k^R(X,Y)$ the set of formal R-linear combinations of elementary finite correspondences. For example when $R = \mathbb{Z}$, we have that correspondences are particular kinds of algebraic cycles:

$$\mathrm{Corr}_k^R(X,Y) \subseteq Z_k(X \times_k Y).$$

Proposition 2.3. We can construct a category $\mathrm{Corr}(\mathrm{Sm}_k; R)$ with objects $X \in \mathrm{Sm}_k$, with hom-sets given by isomorphism classes of R-linear correspondence, with composition given by pullback.

There is a functor

$$\gamma \colon \mathrm{Sm}_k \to \mathrm{Corr}(\mathrm{Sm}_k; R)$$

$$(X \to Y) \mapsto (X \xleftarrow{\mathrm{id}} X \to Y).$$

We claim that $\mathrm{Corr}(\mathrm{Sm}_k; R)$ admits a monoidal structure for which γ is symmetric monoidal (the symmetric monoidal structure on Sm_k is given by cartesian product).

Notation 2.4. If $\mathscr{C}$ is a small category with finite coproducts, we denote by $\mathrm{PSh}_\Sigma \subseteq \mathrm{PSh}$ the full subcategory spanned by presheaves sending finite coproducts to products. For example because disjoint unions are Nisnevich covers, we have that

$$\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k) \subseteq \mathrm{PSh}_\Sigma(\mathrm{Sm}_k).$$

Definition 2.5. An *R-linear Σ-presheaf with transfers* is an object of

$$\mathrm{PSh}_\Sigma(\mathrm{Corr}(\mathrm{Sm}_k; R)).$$

Alternatively, instead of looking at presheaves valued in spaces, we can look at additive R-linear presheaves valued in R-modules. This gives us the category of *R-linear presheaves with transfers*:

$$\mathrm{PST}(k; R) := \mathrm{Fun}_R^{\oplus}\left(\mathrm{Corr}(\mathrm{Sm}_k; R)^{\mathrm{op}}, \mathrm{Mod}_R\right).$$

This latter category is abelian with enough injectives (c.f. [MVW06, p. 2.3]).

There are now two paths forward to define motivic cohomology

(1) Pretend $\mathrm{PSh}_\Sigma(\mathrm{Corr}(\mathrm{Sm}_k; R))$ is $\mathrm{PSh}(\mathrm{Sm}_k)$ and mirror the construction of the stable motivic homotopy category (localize at $\mathbb{A}^1$, sheafify, invert the projective line). This gives us a category of highly structured motivic spectra, called the category of *Voevodsky motives*. It comes with natural adjunctions to ordinary motivic spectra, and we will construct the spectrum representing motivic cohomology via this adjunction.
(2) Consider the derived category of the abelian category $\mathrm{PST}(k; R)$, and construct *motivic complexes* whose hypercohomology compute motivic cohomology.

It's a powerful result that these agree. The former gives us powerful formal properties like an E_∞-structure on motivic cohomology, while the latter gives us explicit formulas to carry out computations. We will give an overview of both.

3. Motivic cohomology, the fancy way

The inclusion $\gamma\colon \mathrm{Sm}_k \to \mathrm{Corr}(\mathrm{Sm}_k^{\mathrm{op}}, R)$ induces an adjunction

$$R_{\mathrm{tr}}\colon \mathrm{PSh}_\Sigma(\mathrm{Sm}_k) \rightleftarrows \mathrm{PSh}_\Sigma(\mathrm{Corr}(\mathrm{Sm}_k; R)) : \gamma^* \tag{1}$$

On the left hand side we localize at $\mathbb{A}^1$ and Nisnevich sheafify. We can do the analogous process on the right hand side and we get a category of *R-linear motivic spaces with transfers*, denoted $\mathrm{Spc}^{\mathrm{tr}}(k; R)$:

$$\begin{tikzcd}
\mathrm{Sm}_k \arrow[r] & \mathrm{PSh}_\Sigma(\mathrm{Sm}_k) \arrow[r, shift left] \arrow[d, shift right] & \mathrm{Spc}(k) \arrow[l, shift left] \arrow[r, shift left] & \mathrm{Spc}(k)_* \arrow[l, shift left] \arrow[r, shift left] \arrow[d, shift right] & \mathrm{SH}(k) \arrow[l, shift left] \\
\mathrm{Corr}(\mathrm{Sm}_k) \arrow[r] & \mathrm{PSh}_\Sigma(\mathrm{Sm}_k) \arrow[u, shift right] \arrow[rr, shift left] & & \mathrm{Spc}^{\mathrm{tr}}(k; R) \arrow[ll, shift left] \arrow[u, shift right] &
\end{tikzcd}$$

Theorem 3.1. Working over a perfect field, there is no need to iterate $\mathbb{A}^1$-localization and Nisnevich sheafification here — that is, we have that

$$L_{\mathrm{mot}} = L_{\mathrm{Nis}} L_{\mathbb{A}^1}\colon \mathrm{Pre}_\Sigma^{\mathrm{tr}}(\mathrm{Sm}_k; R) \to \mathrm{Spc}^{\mathrm{tr}}(k).$$

Proposition 3.2. The category $\mathrm{PSh}_\Sigma^{\mathrm{tr}}(\mathrm{Sm}_k; R)$ is obtained as a localization of the category of chain complexes of R-linear functors

$$\mathrm{Ch}_{\geq 0}\left(\mathrm{Fun}_R(\mathrm{Corr}(\mathrm{Sm}_k; R)^{\mathrm{op}}, \mathrm{Mod}_R\right),$$

where Mod_R is the ordinary 1-category of R-modules.

Corollary 3.3. Every object in $\mathrm{PSh}^{\mathrm{tr}}_{\Sigma}(\mathrm{Sm}_k; R)$ can be modeled by an honest chain complex of presheaves of R-modules

Remark 3.4. The adjunction in Equation 1 descends to an adjunction (we use the same notation for functors by abuse of notation): There is a natural functor

$$R_{\mathrm{tr}} \colon \mathrm{Spc}(k)_* \rightleftarrows \mathrm{Spc}^{\mathrm{tr}}(k; R) : \gamma^*.$$

Here $\mathrm{Spc}^{\mathrm{tr}}(k; R)$ is an R-linear version of the category $\mathrm{DM}^{\mathrm{eff}}_{\geq 0}(k)$ of effective bounded below Voevodsky motives.

Notation 3.5. We denote by $R(q)[p]$ the image of the (p, q)-sphere under R_{tr}:

$$R_{\mathrm{tr}}(S^{p,q}) =: R(q)[p].$$

We refer to $R(1)[2]$ as the *Tate motive.*

By inverting the Tate motive $R(1)[2] = R_{\mathrm{tr}}(\mathbb{P}^1)$, we obtain the category of *motivic spectra with transfers*:

$$\mathrm{SH}^{\mathrm{tr}}(k; R) := \mathrm{Spc}^{\mathrm{tr}}(k; R)\left[R_{\mathrm{tr}}(\mathbb{P}^1)^{-1}\right].$$

This is what's often denoted by $\mathrm{DM}(k)$, the category of *Voevodsky motives.*

Remark 3.6. The notation $\mathrm{DM}(k)$ is intended to mimic $D(M(k))$ – this is intended to be the derived category of the hypothetical abelian category of motives.

By formal nonsense, γ further induces an adjunction

$$R_{\mathrm{tr}} \colon \mathrm{SH}^{\mathrm{tr}}(k; R) \rightleftarrows \mathrm{SH}(k) : \gamma^*.$$

Here R_{tr} is symmetric monoidal, and hence γ^* is lax monoidal.

Theorem 3.7. (Voevodsky cancellation) If k is a perfect field, then the stabilization map

$$\Sigma^{\infty}_{\mathrm{tr}} \colon \mathrm{Spc}^{\mathrm{tr}}(k; R) \to \mathrm{SH}^{\mathrm{tr}}(k; R)$$

is fully faithful.

Upshot 3.8. Any spectrum lying in the image of this embedding can be modeled as an explicit chain complex of presheaves with transfers.

We remark the only algebraic thing we did was look at correspondences. The rest was formal categorical nonsense. Nevertheless, we get a definition of motivic cohomology out of this nonsense.

Definition 3.9. We define *motivic cohomology* $HR^{\mathrm{mot}} \in \mathrm{SH}(k)$ to be

$$HR^{\mathrm{mot}} := \gamma^* R_{\mathrm{tr}}(S^0).$$

Remark 3.10. Since R_{tr} is symmetric monoidal, γ^* is lax monoidal and hence preserves CAlg objects. Therefore HR^{mot} has an E_∞ structure for free.

Notation 3.11. We denote by

$$HR^{p,q}(X) = H^p_{\mathrm{mot}}(X, R(q)) := \pi_0 \mathrm{Map}_{\mathrm{SH}(k)}\left(\Sigma^\infty X_+, \Sigma^{p,q} HR^{\mathrm{mot}}\right).$$

So motivic cohomology is representable by definition.

4. Motivic cohomology, the non-fancy way

Recall we had this category of *R-linear presheaves with transfer*:

$$\mathrm{PST}(k;R) := \mathrm{Fun}^{\oplus}_R \left(\mathrm{Corr}(\mathrm{Sm}_k;R)^{\mathrm{op}}, \mathrm{Mod}_R\right).$$

Simplification: It's actually totally okay to carry everything out over $\mathbb{Z}$ and then tensor with R at the very end, so we'll do that simplification here, to get the $\mathbb{Z}$-linear category of presheaves with transfers that people may be familiar with.

$$\mathrm{PST}(k) := \mathrm{Fun}^{\oplus}\left(\mathrm{Corr}(\mathrm{Sm}_k)^{\mathrm{op}}, \mathrm{Ab}\right).$$

We'd still like to impose some sort of $\mathbb{A}^1$-invariance here, and we know how to do this, via the singular construction:

$$\begin{aligned} C_\bullet \colon \mathrm{PST}(k) &\to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{PST}(k)) \\ F &\mapsto F(U \times \Delta^\bullet). \end{aligned}$$

By taking an alternating sum of face maps (i.e. Dold–Kan), we get a chain complex which we also call $C_\bullet F$ by abuse of notation:

$$\begin{aligned} C_\bullet \colon \mathrm{PST}(k) &\to \mathrm{Ch}_{\geq 0}(\mathrm{PST}(k)) \\ F &\mapsto C_\bullet F. \end{aligned}$$

Proposition 4.1. [MVW06, p. 2.19] The homology presheaves $H_n C_\bullet F$ are homotopy invariant for any n and any $F \in \mathrm{PST}(k)$.

Remark 4.2. (On hypercohomology) The idea of *hypercohomology* is to extend the allowable coefficients for sheaf cohomology from a single sheaf of abelian groups to a *complex* of abelian groups. It's *almost* the same as cohomology with coefficients in the cohomology sheaves of the complex, in the sense that there is a spectral sequence for $F \in \mathrm{Ch}(\mathrm{Ab}(X))$:

$$H^p(X, H^q(F^\bullet)) \Rightarrow \mathbb{H}^{p+q}(X, F^\bullet).$$

We now have a composite functor[1]

$$\mathrm{Sm}_k^{\mathrm{op}} \xrightarrow{\mathbb{Z}_{\mathrm{tr}}} \mathrm{Corr}(\mathrm{Sm}_k;R)^{\mathrm{op}} \hookrightarrow \mathrm{PST}(k) \xrightarrow{C_\bullet} \mathrm{Ch}_{\geq 0}(\mathrm{PST}(k)).$$

We can mirror the smash product construction without creating motivic spaces by taking two pointed schemes (X,x) and (Y,y) and defining

$$\mathbb{Z}_{\mathrm{tr}}(X \wedge Y) := \mathrm{coker}\left(\mathbb{Z}_{\mathrm{tr}}(X) \oplus \mathbb{Z}_{\mathrm{tr}}(Y) \xrightarrow{\mathrm{id}\times y + x \times \mathrm{id}} \mathbb{Z}_{\mathrm{tr}}(X \times Y)\right).$$

Definition 4.3. We define the *motivic complex* $\mathbb{Z}(q)$ to be

$$\mathbb{Z}(q) := C_\bullet \mathbb{Z}_{\mathrm{tr}}(\mathbb{G}_m^{\wedge q})[-q].$$

It turns out we already get some kind of strong sheaf condition without sheafifying!

Proposition 4.4. [MVW06, pp. 3.2, 6.2, 6.4] For every $Y \in \mathrm{Sm}_k$, we have that $\mathbb{Z}_{\mathrm{tr}}(Y)$ is an étale sheaf and $C_\bullet \mathbb{Z}_{\mathrm{tr}}(Y)$ is a chain complex of étale sheaves.

Notation 4.5. For any ring R (or abelian group) we denote by $R(q) := \mathbb{Z}(q) \otimes R$ the (derived?) tensor product.

[1] The notation between [Hoy15] and [MVW06] unfortunately gets swapped here. We should probably call this γ but instead we're calling it $\mathbb{Z}_{\mathrm{tr}}$ to match with what's written in [MVW06].

Definition 4.6. We define *motivic cohomology* to be hypercohomology with coefficients in the motivic complex

$$H^{p,q}_{\mathrm{mot}}(X,R) := \mathbb{H}^p_{\mathrm{Zar}}(X,R(q)).$$

We can compute motivic cohomology as hypercohomology in either the Zariski or Nisnevich site, it doesn't matter which.

Proposition 4.7. [MVW06, pp. 13.9, 13.10, 13.11] Let k be a perfect field, and let $X \in \mathrm{Sm}_k$ be smooth. Then motivic cohomology can be computed using Nisnevich hypercohomology

$$H^{p,q}_{\mathrm{mot}}(X;R) = \mathbb{H}^p_{\mathrm{Zar}}(X,R(q)) = \mathbb{H}^p_{\mathrm{Nis}}(X,R(q)).$$

Warning 4.8. Hypercohomology in the étale site doesn't agree in general. It does in a range! This is essentially the content of the Bloch–Kato conjecture.

Bloch–Kato: Let X be smooth over k, and ℓ invertible in k. Then

$$H^p_{\mathrm{mot}}(X,\mathbb{Z}/\ell(q)) \to H^p_{\mathrm{et}}(X,\mathbb{Z}/\ell(q))$$

is an isomorphism for $q \geq p$ and injective for $q \geq p-1$.

5. Some computations

Example 5.1. [MVW06, p. 21]

(1) $R(q) = 0$ for $q < 0$, by convention
(2) $R(0) = R$ concentrated in degree zero
(3) $\mathbb{Z}(1) = \mathbb{G}_m[-1]$ [MVW06, p. 4.1]

So

$$H^{p,q}(X;R) = 0 \qquad \text{for } q < 0.$$

For X irreducible:

$$H^p(X,R(0)) = H^p(X,R).$$

In weight one, we get

$$\begin{aligned} H^0(X,\mathbb{Z}(1)) &= 0 \\ H^1(X,\mathbb{Z}(1)) &= \mathscr{O}^\times(X) \\ H^2(X,\mathbb{Z}(1)) &= \mathrm{Pic}(X). \end{aligned}$$

The computation of weight one motivic cohomology with $\mathbb{Z}/\ell$ coefficients is a bit more subtle. Without using the full strength of Bloch–Kato we can still see that [MVW06, p. 4.8]

$$\mathbb{Z}/\ell(1)_{\mathrm{et}} \simeq \mu_\ell,$$

which lets us compute

$$H^1(X;\mathbb{Z}/\ell(1)) \cong H^1_{\mathrm{et}}(X,\mu_\ell)$$

by a combination of Hilbert 90 and some diagram chasing ([MVW06, p. 4.9]).

This also can be proven by arguing that

$$\mathbb{Z}/\ell(q) = \mathbb{G}_m/q\mathbb{G}_m[-1]$$

over nice bases (c.f. [Elm, p. 2.0.5]).

The motivic cohomology groups agree with Bloch's higher Chow groups:

Theorem 5.2. (Voevodsky) Let X be a smooth variety. Then we have

$$H^p(X;\mathbb{Z}(q)) = H^{p,q}_{\mathrm{mot}}(X;\mathbb{Z}) = \begin{cases} \mathrm{CH}^q(X, 2q-p) & q \geq 0 \text{ and } 2q-p \geq 0 \\ 0 & \text{else} \end{cases}$$

c.f. [Hai, p. 4.10].

Example 5.3. When $2q = p$, we obtain the ordinary Chow groups

$$H^{2q,q}(X;A) \cong \mathrm{CH}^q(X) \otimes A.$$

We have seen motivic cohomology vanishes for $q \leq 0$ (actually $q \leq 1$). We establish some further vanishing results.

Theorem 5.4. (Vanishing theorems, [MVW06, pp. 3.6, 19.3]) If X is smooth, then

$$H^{p,q}(X, A) = 0$$

for $p > q + \dim X$ or $p > 2q$

Proof. The first range is immediate. We have that $\mathbb{Z}(q)$ is zero in degrees above q. Since $H^i_{\mathrm{Zar}}(X, -)$ vanishes for $i > \dim X$, so vanishing for $p > q + \dim X$ follows by the hypercohomology spectral sequence. Vanishing for the other range is a bit harder. ☐

6. Eilenberg–MacLane spaces

Since we have defined $HR \in \mathrm{SH}(k)$, we can obtain *Eilenberg–MacLane spaces* in the following way:

$$K(R(q), p) := \Omega^\infty HR \otimes S^{p,q}.$$

Note that $S^{p,q}$ is only a space when $p \geq q$ and $p > 0$.[2] We can define Eilenberg–MacLane spaces outside this range by taking loop spaces.

Warning: For any sheaf of abelian groups $\mathcal{A}$, we can construct Eilenberg–MacLane spaces $K(\mathcal{A}, n)$, representing $H^n_{\mathrm{Nis}}(-,\mathcal{A})$, with homotopy sheaves concentrated in a single simplicial degree. The spaces $K(R(q), p)$ do *not* have this property. They are $(p-1)$-connected, but have higher nontrivial homotopy sheaves (see e.g. [AFH19, p. 2.21])

Example 6.1. Since $\mathbb{Z}(1) = \mathscr{O}^\times[-1]$, we get that

$$\mathbb{G}_m = K(\mathbb{Z}(1), 1).$$

The Picard group of line bundles is $H^{2,1}(-,\mathbb{Z})$ by comparison to Chow, so we get

$$\mathbb{P}^\infty = K(\mathbb{Z}(1), 2).$$

Remark 6.2. We are particularly interested in the Eilenberg–MacLane spaces $K(\mathbb{Z}(2n), n)$, since these represent Chow groups.

[2] In [Voe10, p. 6] Voevodsky remarks it is hard to describe the structure of $K(A(q), p)$ (he calls them $K(A, p, q)$) when $p < q$ and $q \geq 2$.

7. Motivic cohomology of an algebraically closed field

As an example we will use in the seminar, we want to compute $H^{*,*}_{\text{mot}}(\text{Spec}\,(k);\mathbb{Z}/\ell)$ for k an algebraically closed field.

Multiplication by ℓ induces a map

$$\mathbb{G}_m \xrightarrow{\ell} \mathbb{G}_m,$$

whose fiber is a $K(\mathbb{Z}/\ell(1),0)$. A choice of primitive ℓth root of unity, given by $S^0 \to \mathbb{G}_m$, gives rise to a class we call $\tau \in \pi_0 K(\mathbb{Z}/\ell(1),0)$:

$$\begin{tikzcd} & S^0 \arrow[dl, dashed] \arrow[d] & \\ K(\mathbb{Z}/\ell(1),0) \arrow[r] & \mathbb{G}_m \arrow[r, "\ell"'] & \mathbb{G}_m. \end{tikzcd}$$

In particular we obtain some class $\tau \in H^{0,1}_{\text{mot}}(k;\mathbb{Z}/\ell)$.

Proposition 7.1. If $k = \bar{k}$, there is an isomorphism

$$H^*_{\text{et}}(\text{Spec}\,(k);\mu_\ell) = \begin{cases} \mathbb{Z}/\ell & * = 0 \\ 0 & \text{else} \end{cases}$$

Via comparison to the étale site, and using our truncation result above, we can conclude the following theorem of Suslin.

Theorem 7.2. [Voe10, p. 83] If $k = \bar{k}$, we have that

$$H^{*,*}_{\text{mot}}(\text{Spec}\,(k);\mathbb{Z}/\ell) \cong \mathbb{Z}/\ell[\tau].$$

Here $|\tau| = (0,1)$.

Proof sketch. We want to compute $H^p(k,\mathbb{Z}/\ell(q))$. Since $\mathbb{Z}(q) = 0$ for $q < 0$, we assume $q \geq 0$. Since a field is dimension zero, by the vanishing theorem, we have that $H^p(k,\mathbb{Z}/\ell(q)) = 0$ for $p > q$. Hence we're left with the range $p \leq q$, which is precisely the range in which Bloch–Kato implies we are isomorphic to étale cohomology.

Since $\mathbb{Z}/\ell(q)_{\text{et}} = \mu_\ell^{\otimes q} = \mu_\ell$ for any q (by algebraic closure), we conclude that

$$H^{p,q}(k,\mathbb{Z}/\ell) = \begin{cases} \mathbb{Z}/\ell & p = 0, q \geq 0 \\ 0 & \text{else} \end{cases}$$

In particular the result follows. □

Remark 7.3. We have that $H^0_{\text{et}}(X,\mathbb{Z}/\ell(q)) = \mathbb{Z}/\ell$ even when q is negative. So we have that

$$H^*_{\text{et}}(k,\mathbb{Z}/\ell(*)) = \mathbb{Z}/\ell[\tau,\tau^{-1}].$$

Corollary 7.4. Let X be any motivic space or spectrum. Then $H^{*,*}(X,\mathbb{Z}/\ell)$ is a module over $\mathbb{Z}/\ell[\tau]$.

8. Motives

For any ring R, we can now create the category $\text{Mod}_R := \text{Mod}_{HR^{\text{mot}}}$ of modules over motivic cohomology with R coefficients.

Terminology 8.1. Given $X \in \text{SH}(k)$ a motivic spectrum, we denote by $X \otimes R$ the *motive* attached to X. This is an element in Mod_R.

Definition 8.2. We say an R-module M is *split* if we have an equivalence

$$M \xrightarrow{\sim} \vee_\alpha \Sigma^{p_\alpha, q_\alpha} R,$$

for some (uniquely determined) family of bidegrees (p_α, q_α) (c.f. [Hoy15, p. 193]).

Corollary 8.3. There is a monoidal adjunction

$$\mathrm{Mod}_{HR} \rightleftarrows \mathrm{SH}^{\mathrm{tr}}(k; R),$$

compatible with the adjunction $\mathrm{SH}^{\mathrm{tr}}(k; R) \rightleftarrows \mathrm{SH}(k)$.

Definition 8.4.

(1) An HR-module is *split/cellular* if it is generated by HR-modules of the form $\Sigma^{p,q} HR$.
(2) An object in $\mathrm{SH}^{\mathrm{tr}}(k; R)$ is *cellular* if it is generated by objects of the form $R_{\mathrm{tr}} \Sigma^{p,q} 1$.

Lemma 8.5. [Hoy15, p. 4.4] This restricts to an equivalence on the full subcategories of cellular objects.

Definition 8.6. [Voe10, p. 2.60], [Hoy15, p. 196] An object in $\mathrm{Spc}^{\mathrm{tr}}(k; R)$ is called *split proper Tate* if it is equivalent to a direct sum of objects of the form

$$R_{\mathrm{tr}}(S^{p,q}),$$

for $p \geq 2q$. That is, non-negative S^1-suspensions of tensor powers of $\mathbb{P}^1$.

Theorem 8.7. [Voe10, p. 3.33] We have that

$$(\mathbb{Z}/\ell)_{\mathrm{tr}} K(\mathbb{Z}/\ell(q), p)$$

is split proper Tate for any $p \geq 2q$.

Corollary 8.8. The motivic cohomology

$$H^{*,*}(K(\mathbb{Z}/\ell(a), b), \mathbb{Z}/\ell)$$

is τ-torsion free as a module over $H^{*,*}(k, \mathbb{Z}/\ell)$

Corollary 8.9. The motivic Steenrod algebra at a prime p can be computed as

$$\mathcal{A}^{*,*} = \lim_{n \to \infty} \widetilde{H}^{*+2n, *+n}(K(\mathbb{F}_p(2n), n); \mathbb{F}_p).$$

Sketch. We have to argue a certain $\lim^1$ term vanishes, which simplifies by leveraging that $K(\mathbb{F}_p(2n), n)$ is split proper Tate. ☐

This lets us leverage unstable cohomology operations (reduced power operations and Bocksteins) to describe the motivic Steenrod algebra.

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