SETS, GROUPS, AND GEOMETRY
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THOMAS BRAZELTON

Abstract. Course notes for MATH101: Sets, groups and geometry, taught at Harvard in spring 2025.

0. About

Course notes for MATH101, *Sets, groups and geometry*, taught spring 2025 at Harvard. The goal for this class is to introduce students to the basics of proof-based reasoning. The amount of rigor throughout is a bit variable – we neglect to do certain things in full detail (construction of the real numbers for instance) in order to allocate our time towards other results and ideas. The course is divided into three parts: set theory, group theory, and geometry. The set theory portion followed selections in Richard Hammack's *Book of proof*, 3rd ed. [Ham18]. For algebra, we followed parts of W. Keith Nicholson's *Abstract algebra*, 4th ed. [Nic12]. For the geometry portion we have focused on the construction of polyhedra in three dimensions, and the classification of Platonic and Archimedean solids. An outline of the course is below:

(1) Week 1: Sets and a bit of pure set theory and ZFC
(2) Week 2: Logic and direct proof
(3) Week 3: Contrapositive and contradiction
(4) Week 4: Modular arithmetic and induction
(5) Week 5: Midterm 1
(6) Week 6: Functions, bijections, and permutations
(7) Week 7: Groups, monoids, and isomorphisms
(8) Week 8: Subgroups, group presentations, orders of elements
(9) Week 9: Dihedral groups, Cayley's theorem
(10) Week 10: Midterm 2
(11) Week 11: The orbit-stabilizer theorem, Platonic solids, and their symmetry groups
(12) Week 12: Convex polyhedra
(13) Week 13: Euler's formula

1. Sets

A *set* is a collection of things, and these things are called elements. We won't give a formal definition of a set, since this gets us too deep into mathematical logic, so we'll kind of take a set as a given and build mathematics on top of it.

We denote by $\{1, 2, 3\}$ the set whose elements are the numbers 1, 2, and 3. These curly braces are used to list the elements of a set.

Example 1.1. The set

$$S = \{a, b, c, d\}$$

is a set consisting of four elements, which are *letters* a, b, c, and d.

Note 1.2. Elements are not allowed to be repeated! For instance, $\{a, b, a, c, d\}$ is not a valid set.[1]

Notation 1.3. We use the symbol $\in$ to denote if an element is in a set. So if $T = \{0, 4, 1, 6\}$, we might write

$$1 \in T$$

to mean that 1 is an element of T. We will write $\notin$ to say something is **not** an element of a set. So for instance

$$2 \notin T.$$

Example 1.4. We denote by $\mathbb{N}$ the set of all *natural numbers*, meaning counting numbers including zero:

$$\mathbb{N} = \{0, 1, 2, 3, 4, \ldots\}.$$

We denote by $\mathbb{Z}$ the set of all *integers*[2] meaning all positive and negative counting numbers:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

We denote by $\mathbb{Q}$ the set of all *rational numbers*, meaning numbers of the form $\frac{p}{q}$ where p and q are integers, and $q \neq 0$.

Example 1.5. We don't just need to have numbers and letters be elements of sets. We can really let *anything* be an element in a set. For instance we can take a set of some shapes

$$S = \{\bigcirc, \triangle, \square\}.$$

We can also have *sets* being elements of sets. For instance we can take

$$B = \{\mathbb{N}, \mathbb{Z}, 3, \{4\}\}.$$

This is a set with four elements – the set of natural numbers, the set of integers, the number 3, and the set with one element which is the number 4. This might feel weird but we'll get used to it soon enough.

In the above examples, we didn't list out every element of a set when we wrote it, instead we did a ... when the pattern became clear. For instance what is the following set:

$$A = \{0, 3, 6, 9, 12, 15, \ldots\}.$$

It is the set of all multiples of three! Instead of listing it out, we might *build it*, meaning give a rule for elements to be a part of it. This is done using set builder notation:

$$A = \{3n \colon n \in \mathbb{N}\}.$$

This means A is the set of all numbers of the form $3n$ where n is an element of $\mathbb{N}$.[3]

A special set is the *empty set*, which has no elements. We could write it as $\{\}$ if we wanted, but we use special notation for it, namely $\varnothing$.

[1] This is a convention that we're not allowing for repeated elements. We can build a different type of set theory where you *can* have repeated elements in sets, these are called *multisets*. The math that you build with these becomes a lot more complicated though.

[2] This letter comes from the German *Zahlen*, meaning "numbers."

[3] People who know a little CS, we might think about this as an infinite for loop (for all $n \in \mathbb{N}$, add $3 \cdot n$ to the set we're building, and let A be the resulting output). Obviously this wouldn't terminate on a computer, but we're mathematicians so we can let things happen infinitely many times and keep moving!

1.1. **Cardinality.** If A is a set, we denote by $|A|$ the *cardinality* of the set, roughly meaning its size. It is the number of elements in the set, possibly infinite.

Example 1.6. The cardinality of some sets we've discussed are:

$$\begin{aligned} |\{a,b,c,d\}| &= 4 \\ |\mathbb{N}| &= \infty \\ |\mathbb{Z}| &= \infty \\ |\{\bigcirc, \triangle, \square\}| &= 3 \\ |\{\mathbb{N}, \mathbb{Z}, 3, \{4\}\}| &= 4 \\ |\varnothing| &= 0. \end{aligned}$$

1.2. **Subsets.** Note that every element in $\mathbb{N}$ is an element of $\mathbb{Z}$. When this happens, we write $\subseteq$, and we say one set is a *subset* of the other.

Definition 1.7. Given two sets A and B, we write $A \subseteq B$ if $x \in A$ implies that $x \in B$. In words, every element in A is also an element in B. We write $A \subsetneq B$ if A is *not* a subset of B.

Example 1.8. We have that $\mathbb{N} \subseteq \mathbb{Z}$.

Question 1.9. Given two sets A and B, how would you argue that A is *not* a subset of B?

You just have to find some element in A that is not in B.

Example 1.10. To argue that $A = \{3,6,8,1\}$ is not a subset of $B = \{2,6,8,1,5\}$, we see that $3 \in A$ but $3 \notin B$. Therefore $A \subsetneq B$.

Example 1.11. Let $A = \{1,2,3\}$. Is it true that $\varnothing \subseteq A$?

Yes! The condition that $\varnothing \subseteq A$ means that for every $x \in \varnothing$ we have that $x \in A$. Since $\varnothing$ has no elements, this is true.[4] In fact $\varnothing \subseteq S$ for *any* set S.

1.3. **Set equality.**

Question 1.12. What does it mean for two sets to be equal?

Example 1.13. We claim that $\{4,1,0\} = \{0,1,4\}$.

Answer 1.14. Two sets A and B are equal if they have the same elements. Phrased differently, $x \in A$ implies $x \in B$ and $x \in B$ implies $x \in A$. That is, $A \subseteq B$ and $B \subseteq A$.

1.4. **Operations with sets.** Given two sets A and B we denote by $A \cup B$ their *union*, meaning the set of all elements in A or in B.

$$A \cup B = \{x \colon x \in A \text{ or } x \in B\}.$$

Example 1.15. We have that

$$\{1,2,3\} \cup \{4,5,6\} = \{1,2,3,4,5,6\}.$$

Note we don't allow repeats, so

$$\{1,2,3\} \cup \{3,4\} = \{1,2,3,4\}.$$

[4] We refer to statements like this as *vacuously true* – they're true because no elements exist to check the conditions on. For example we might say "every number which is both even and odd is equal to 7." This is a true statement, not because 7 is both even and odd, but because no numbers are both even and odd.

Definition 1.16. Given two sets A and B, we denote by $A \cap B$ their *intersection*, meaning the set of all elements in both A **and** B:

$$A \cap B = \{x \colon x \in A \text{ and } x \in B\}.$$

Example 1.17. We have

$$\{1, 2, 3, 4\} \cap \{3, 4, 5, 6\} = \{3, 4\}.$$

Question 1.18. What is

$$\{1, 2, 3\} \cap \{4, 5, 6\}?$$

It is the empty set! There are no elements in both sets.

Finally we denote by $A - B$ their *difference*, meaning

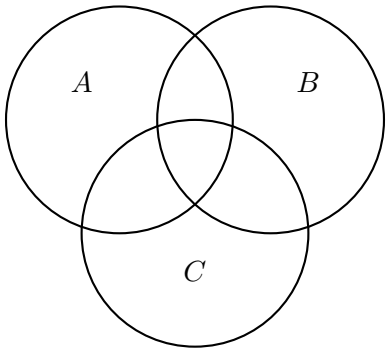
$$A - B = \{x \colon x \in A \text{ and } x \notin B\}.$$

For instance

$$\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}.$$

Note that difference depends on the order of sets! We always have that $A \cup B = B \cup A$ and $A \cap B = B \cap A$, but $A - B$ and $B - A$ might be different sets.

Venn diagrams are a great way to visualize sets and their overlaps:



1.5. **Power sets.** Given a set A, we denote by

$$\mathcal{P}(A) := \{X \colon X \subseteq A\}$$

the *power set* of A, meaning the set of all subsets of A.

Question 1.19. What is the power set of $\{1, 2\}$?

It is the set

$$\mathcal{P}(\{1, 2\}) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}.$$

Don't forget that $\varnothing \subseteq S$ and $S \subseteq S$ for every set S.

Question 1.20. If S has cardinality n, what is the cardinality of the power set $\mathcal{P}(S)$?

1.6. **The real numbers.** We denote by $\mathbb{R}$ the set of *real numbers*. These are numbers we think about as lying on the number line, but need not be rational. For instance $\pi \in \mathbb{R}$ but $\pi \notin \mathbb{Q}$.[5] It's not super easy to define $\mathbb{R}$ formally, so we'll avoid doing this here.

We define *intervals* to be subsets of $\mathbb{R}$. You may have seen the notation $[0, 1]$ before. This refers to the *closed interval* between zero and one. Explicitly in terms of set builder notation, we would write:

$$[0, 1] = \{x \in \mathbb{R} \colon 0 \leq x \text{ and } x \leq 1\}.$$

[5]This is not super easy to prove, but we'll see examples later of irrational numbers.

We also have open intervals, denoted by (a, b). For instance

$$(2, 3) := \{x \in \mathbb{R} \colon 2 < x \text{ and } x < 3\}.$$

Exercise 1.21. Let $A = \{a, b, c, d, e\}$, let $B = \{d, e, f\}$, and let $C = \{1, 2, 3, d\}$. Write out the following

(1) $A \cup B$
(2) $A \cap B$
(3) $A - B$
(4) $B - A$
(5) $(A - B) \cup (B - A)$
(6) $A \cap C$
(7) $B \cap C$
(8) $A \cup (B \cap C)$.

Draw out a Venn diagram and check your answers!

Exercise 1.22. Using the set $\mathbb{Z}$, define the set $\mathbb{Q}$ of rational numbers using set builder notation.

Exercise 1.23. Let $S = \{a, b, c\}$.

(1) Write out all the elements of the power set $\mathcal{P}(S)$.
(2) What is the cardinality $|\mathcal{P}(S)|$?

Exercise 1.24. Let $S = \{5a + 2b : a, b \in \mathbb{Z}\}$.

(1) Is it true that $S \subseteq \mathbb{Z}$? Why or why not?
(2) Is it true that $S \subseteq \mathbb{N}$? Why or why not?
(3) Is it true that $\mathbb{Z} = S$? Why or why not?

1.7. **Cartesian products.**

Definition 1.25. An *ordered pair* is a tuple of two things (x, y).

Definition 1.26. Given two sets A and B, we define their *(Cartesian) product* denoted $A \times B$ by

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Example 1.27. If $A = \{x, y, z\}$ and $B = \{1, 2\}$ then

$$A \times B = \{(x, 1), (x, 2), (y, 1), (y, 2), (z, 1), (z, 2)\}.$$

Notation 1.28. We write A^2 for the Cartesian product $A \times A$.

Example 1.29. When we graph things on the xy-plane, we are thinking about a *subset* of $\mathbb{R}^2$

$$\mathbb{R}^2 = \{(x, y) : x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}.$$

In particular if $y = f(x)$ is a function, we could graph the subset

$$\big\{(x, y) \in \mathbb{R}^2 : y = f(x)\big\} \subseteq \mathbb{R}^2.$$

This is most of what we do in high school algebra - studying subsets of $\mathbb{R}^2$ of this form.

Observe: For any two finite sets A and B, we have that

$$|A \times B| = |A| \cdot |B|.$$

We can iterate multiplying sets, for instance if $A_1, A_2, \ldots, A_n$ are all sets, then we denote by

$$A_1 \times \cdots \times A_n = \{(x_1, \ldots, x_n) : x_i \in A_i \text{ for each } i = 1, \ldots, n\}.$$

We might use shorthand notation for this:

$$\prod_{i=1}^{n} A_i = A_1 \times \cdots \times A_n.$$

This type of notation is common also for unions and intersections of more than two sets:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup \cdots \cup A_n$$
$$\bigcap_{i=1}^{n} A_i = A_1 \cap \cdots \cap A_n.$$

1.8. **Index sets.** If we have sets $A_1, A_2, \ldots, A_n$, another way to phrase this is that we have sets *indexed* over the set $I = \{1, 2, \ldots, n\}$. In other words for each $i \in I$ we have a set A_i. In that way we can rewrite the operations above as

$$\prod_{i \in I} A_i, \qquad \bigcup_{i \in I} A_i, \qquad \bigcap_{i \in I} A_i.$$

From this perspective it's not really important that I was a subset of the natural numbers. We can have sets A_i indexed over *any* index set I.

1.9. **Complements.** If $B \subseteq A$ is a subset, we denote by B^c the *complement* of B in A, meaning everything that is in A and not in B:

$$B^c = \{x \in A : x \notin B\}.$$

Note that Hammack writes this as $\bar{B}$.

Exercise 1.30. Discuss:

(1) Let A be a set. Describe the set $A \times \varnothing$. What is its cardinality?
(2) Are $A \times B \times C$ and $(A \times B) \times C$ equal as sets? Why or why not?

Exercise 1.31. Let A be a set.

(1) What is $\varnothing^c$ (here $\varnothing$ is viewed as a subset of A).
(2) Let $B \subseteq A$ and $C \subseteq A$ be subsets. What is $(B \cap C)^c$?
(3) What is $(B \cup C)^c$?

2. Axiomatic rules for sets

We've mentioned that it's hard to define sets, but that they satisfy certain rules. We'll lay these out now. These rules were developed by Zermelo and Fraenkel in the first few decades of the 20th century, building on work in formal logic and set theory in the 19th century. We call these axioms **ZF** after Zermelo and Fraenkel, and there are 8 axioms in total.

Definition 2.1. A set X is a *pure set* or a *hereditary set* if all of its elements are themselves sets, and all of the elements of those sets are sets, and so on.

Example 2.2. The empty set $\varnothing$ is vacuously a pure set. The set $\{\varnothing\}$ or $\{\varnothing, \{\varnothing\}\}$, are also pure, for instance.

Pure set theory: Let's treat this like a game, and temporarily forget everything we're allowed to do with sets. Our pieces are pure sets, and here are the rules.

(1) given any two sets A and B, you are allowed to ask if they are equal, and the answer is either true or false.[6]
(2) given any two sets A and B you're allowed to ask if $A \in B$, and the answer is either true or false.
(3) you're allowed to use as many variables as you want to represent sets
(4) you're allowed to negate any statement and ask if it is true or false (i.e. is it true that $A \neq B$)
(5) you can make "for all" and "implies"statements, like "for all $X \in A$" this "implies" that $X \in B$ (meaning $A \subseteq B$)
(6) you can make "there exists" statements like "there exists $x \in A$ so that $X \notin B$" (meaning $A \not\subseteq B$).

On top of these ground rules we're going to have some *axioms*. An *axiom* is like a mathematical rule. They are some base facts that you take for granted, and build mathematics off of. By no means are the axioms we're laying out here the only axioms you could build mathematics off of, and we're not even necessarily saying they're "true." They just end up leading to a convenient formulation of a lot of things we want to do in math.

Note 2.3. The numbering here is not a standard thing, I'm just using it to keep track of stuff easier.

ZF1: *(Axiom of extensionality)* Two sets are equal if they have the same elements.
ZF2: *(Axiom of union)* Unions of sets exist.[7]
ZF3: *(Axiom of power set)* Power sets exist – if A is a set then $\mathcal{P}(A)$ is a valid set.
ZF4: *(Axiom of pairing)* If A and B are sets, then the set $\{A, B\}$ exists.

Corollary 2.4. If A is a set then $\{A\}$ is a set.

Proof. Since A is a set, we can apply the axiom of pairing to A and itself to form the set $\{A, A\}$. Since sets can't have repeated elements, this set $\{A, A\}$ guaranteed by the axiom of pairing only has *one* element, so we abbreviate it $\{A\}$. □

ZF5: *(Axiom of regularity)* If S is a nonempty set, then it contains an element $T \in S$ so that T and S are disjoint sets (have no elements in common).

This is maybe nonintuitive but it has some important applications.

Corollary 2.5. No set can contain itself as an element.

Proof. Let A be any set, and consider the set $S = \{A\}$. By **ZF5**, S contains an element that is disjoint from itself, and since S only has one element, this implies that S is disjoint from A. In other words A and $\{A\}$ have no elements in common, so in particular $A \notin A$. □

ZF6: *(Axiom schema of specification)* You can build sets with set builder notation.[8]

[6]Just like anything in math, we could ask what happens if we remove some of the basic building blocks. What happens if we let statements like $A = B$ admit another truth value - not true or false but something else? What if, for instance, the *truth* of a statement is a number in the interval $[0, 1]$ where 0 is absolutely false and 1 is absolutely true, but we can have intermediate stages? These kinds of questions lead us to something called *fuzzy logic*, a fascinating detour we sadly won't have time to explore in this class.

[7]The precise statement is if A is a pure set, there exists a set $\cup_{B \in A} B$ which is a union of all the elements of A (the most precise statements says there is a set *containing* $\cup_{B \in A} B$, and we can shorten this to $\cup_{B \in A} B$ using the axiom of pairing). For CS people, this is an axiomatization of the process of *flattening* a set or a list.

[8]We're being vague here – **ZF6** tells you more concretely *what kinds of formulas* you're allowed to use in set builder notation, but let's treat this as a black box for the time being.

Explicitly, **ZF6** says that the following type of set building is allowed:[9]

$$\{x \in A : \text{something about } x \text{ is true}\}.$$

But this type of set building is not allowed:

$$\{x : \text{something about } x \text{ is true}\}.$$

Why can't we let the latter exist?

Russell's paradox: Suppose we're allowed to build sets of that form, and we take

$$S = \{x : x \notin x\}.$$

We've already seen that no set can contain itself, so $x \notin x$ for every set x. In particular S contains *every set*. But S itself is a set, which means $S \in S$. But also $S \notin S$. These can't both be true, so we've broken math!

It's generally advisable not to break math, so we exclude sets built like this. The point is not whether $S \in S$ or whether $S \notin S$, the point is such a set S *cannot be allowed to exist* if we want a logically consistent framework of math.

Barber's paradox (a common application of Russell's paradox): A barber cuts everyone's hair who doesn't cut their own hair. Does the barber cut their own hair?

ZF7: *(Axiom schema of replacement)* The domains of functions are sets (roughly speaking).

ZF8: *(Axiom of infinity)* There exists a set with infinitely many elements.

There is a 9th mysterious axiom, called the *axiom of choice*. This isn't one of the ZF axioms, so when we use it we often refer to **ZFC** which is ZF + Choice. We won't go into this as much in this class, but it will become super important later in proof-based mathematics.

Exercise 2.6. Argue, from the axioms of ZF, that the empty set exists.[10]

Exercise 2.7. Argue from the axioms of ZF that a set containing exactly four elements exists.

Exercise 2.8. Regarding **ZF8** (the axiom of infinity):

(1) Define an operation[11] $S(-)$ which inputs a set and outputs a set via the following formula:

$$S(A) = A \cup \{A\}.$$

Argue that if A is a valid set, then $S(A)$ is a valid set.

(2) What is the cardinality $|S(A)|$ in terms of $|A|$?

(3) Write out $S(\varnothing)$, $S(S(\varnothing))$, and $S(S(S(\varnothing)))$.

The essential idea of **ZF8** is that the set

$$S(S(\cdots S(\varnothing)))$$

exists, where we iterate S infinitely many times.

As a note, this is how von Neumann *constructs* the natural numbers in the language of set theory. You can define $0 = |\varnothing|$ to be the cardinality of the empty set, define $1 = |S(\varnothing)|$, $2 = |S(S(\varnothing))|$, etc. In this language, **ZF8** posits that $\mathbb{N}$ exists as a set.

Exercise 2.9. Let S be the set of all sets. Is S a valid set by the axioms of ZF? Why or why not?

Exercise 2.10. Given two sets A, B, can it be true that $A \in B$ and $B \in A$? If so, give an example. If not, explain why not from the axioms.

[9] We're being intentionally vague with this "something about x." The precise things that are allowed to be here are what are called *first order formulas*. We'll get into these more next week.

[10] This is often taken as an axiom itself, but it can be proven from the other axioms in the way we've laid it out.

[11] Here S stands for "successor."

3. LOGIC

A *statement* is any mathematical sentence that can definitively evaluated as true or false.

Here are some examples of statements:

(1) It is Monday today
(2) The number 2 is even
(3) The number 2 is not even
(4) There exists a finite subset of X.
(5) Every natural number is divisible by a prime number
(6) Every subset of an infinite set is infinite.

We can evaluate each of these as true or false.

Let P be a mathematical statement. Then we can assign it a *truth value* meaning an element of the set $\{T, F\}$ where T stands for true and F stands for false.

We can *negate* mathematical statements, which swaps the truth value of the statement. We denote this new statement by $\neg P$ (Hammack writes $\sim P$)

P	$\neg P$
It is Monday today	It is not Monday today
The number 2 is even	The number 2 is not even
The number 2 is not even	The number 2 is even

Pause – what happened here? Let P be "the number 2 is even." Then we just said

$$\neg\neg P \text{ is the same statement as } P.$$

This is called *double negation elimination*.[12] It's an admissible rule in our logical framework that we can cancel two negation symbols when they appear right next to each other.

Let's keep negating:

P	$\neg P$
There exists a finite subset of X	There does not exist a finite subset of X
or	For every subset of X, it is not finite.

Interesting – when we negate a "there exists" statement, we get a "for every" statement.

Let's keep negating:

P	$\neg P$
Every natural number is divisible by a prime number	Not every natural number is divisible by a prime number
or	There exists a natural number which is not divisible by a prime number
or more nicely written:	There exists a natural number which is not divisible by *any* prime number

Same deal – negating an "every" statement gets us a "there exists" statement. Finally:

P	$\neg P$
Every subset of an infinite set is infinite	Not every subset of an infinite set is infinite.
or	There exists a finite subset of an infinite set.

3.1. "And" and "or".

We can combine statements with the words "and" and "or" to get new statements.

Notation 3.1. Given two statements P and Q we write $P \wedge Q$ to mean "P *and* Q."

[12] There exist frameworks of logic that *explicitly reject this*, but classical logic accepts it and so will we in this class.

Question: How does the truth of $P \wedge Q$ depend on the truth of P and the truth of Q?
Let's consider an example:

$$P = \text{“it is Monday”}$$
$$Q = \text{“it is raining”}.$$

Then

$$P \wedge Q = \text{“it is Monday } \textbf{and} \text{ it is raining”}$$

Let's consider the four possibilities for P being true and false and Q being true and false.

it is Monday	it is raining	$\Rightarrow$	$P \wedge Q$ is true
it is Monday	it is not raining	$\Rightarrow$	$P \wedge Q$ is false
it is not Monday	it is raining	$\Rightarrow$	$P \wedge Q$ is false
it is not Monday	it is not raining	$\Rightarrow$	$P \wedge Q$ is false.

We can encode this more concisely in a *truth table*:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F.

Definition 3.2. If P and Q are statements, we denote by $P \vee Q$ the new statement “P **or** Q.”

Note that $P \vee Q$ will be true if at least one of P and Q are true.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

3.2. **Conditional statements.**

Definition 3.3. Given two statements P and Q, we write $P \Rightarrow Q$ to mean “P **implies** Q.” In other words, whenever P is true, this implies Q must be true as well. We might also phrase this as “if P then Q.”

This one is a little weirder. It's still a statement, so we can input truth values for P and Q and get out a truth value for $P \Rightarrow Q$.

Hammack talks about this as a “promise.” Explicitly, the statement $P \Rightarrow Q$ is the *promise* that any time P is true, then Q will also be true. The truth value of $P \Rightarrow Q$ refers to whether or not the promise was broken.

Example 3.4. Let P be the statement “you pass the exam” and Q be the statement “you pass the class.” The statement $P \Rightarrow Q$ could be a promise the professor makes to the students: “**if** you pass the exam **then** you pass the class.”

In this case truth values of P and Q could be different scenarios that could play out:

you passed the exam	you passed the class	$\Rightarrow$	cool!
you passed the exam	you didn't pass the class	$\Rightarrow$	the promise was broken
you didn't pass the exam	you passed the class	$\Rightarrow$	cool, the promise wasn't broken
you didn't pass the exam	you didn't pass the class	$\Rightarrow$	the promise wasn't broken

We can write this as a truth table

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Definition 3.5. The *converse* of a statement $P \Rightarrow Q$ is the statement $Q \Rightarrow P$. These are *not equivalent statements.*

3.3. **If and only if.**

Definition 3.6. We write $P \Leftrightarrow Q$ to mean P **if and only if** Q. It is a shorthand for $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$. It is a promise that whenever P is true then Q will be true **and** whenever Q is true then P will be true.

We can encode this in a truth table

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Roughly speaking, $P \Leftrightarrow Q$ means that P and Q are the same statement — the truth of one is equivalent to the truth of the other.

3.4. **Quantifiers.**

Definition 3.7. Let X be a set, and let $P(x)$ be a statement (not necessarily true or false) that can be made about any element $x \in X$.

Example 3.8. Let $X = \mathbb{N}$, then $P(x)$ could be any statement you can make about natural numbers, for example "x is even" or "x is prime."

How would you say $P(x)$ is true *for every* $x \in X$? We could try to list out the elements of X in some order, i.e.

$$X = \{x_1, x_2, x_3, \ldots\},$$

and then we could write

$$P(x_1) \wedge P(x_2) \wedge P(x_3) \wedge \cdots$$

This is cumbersome notation, and as we'll soon see we sometimes can't even order the elements in X like that. So we want a shorthand.

Definition 3.9. The symbol $\forall$ means "for all." We use it in the following way:

$$\forall x \in X,\ P(x)$$

this means "for all $x \in X$, $P(x)$ is true."

Similarly, we could make the statement "there exists an $x \in X$ for which $P(x)$ is true." This means either $P(x_1)$ is true, or $P(x_2)$ is true, or $P(x_3)$ is true,... so we could write this as

$$P(x_1) \vee P(x_2) \vee P(x_3) \vee \cdots$$

Again we bump into the same issue that this is cumbersome notation.

Definition 3.10. The symbol $\exists$ means "there exists." In other words

$$\exists x,\ P(x)$$

means "there exists an $x \in X$ for which $P(x)$ is true."

Terminology 3.11. The symbols $\forall$ and $\exists$ are called *quantifiers*.

Example 3.12. We've seen these in calculus — given a function $f\colon \mathbb{R} \to \mathbb{R}$, the statement "$f$ is *continuous* at x_0" is shorthand for the statement

$$\forall \epsilon > 0 \exists \delta > 0 \left(|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon \right).$$

Exercise 3.13. Negate the following phrases

(1) If n is divisible by 10 then n is even
(2) If it is raining then the ground is wet
(3) There exists a nonempty subset of X
(4) n is odd and n is prime

Exercise 3.14. Using truth tables, show that

$$\neg \left(P \wedge Q \right) = (\neg P) \vee (\neg Q)$$

Exercise 3.15. Let A and B be subsets of some larger set C. Let P be the statement $x \in A$ and let Q be the statement $x \in B$. Can you reinterpret Exercise 3.14 in this language? What does this tell you?

Exercise 3.16. Let P and Q be arbitrary statements.

(1) Write out the truth table for $(\neg P) \vee Q$.
(2) Compare this with the truth table for $P \Rightarrow Q$. Explain what you see!

Exercise 3.17. Let $E(n)$ be the statement "n is even" and let $O(n)$ be the statement "n is odd." Translate the following into English sentences:

(1) $E(4)$
(2) $O(k)$
(3) $E(n) \wedge O(n)$
(4) $E(m) \Rightarrow O(m)$
(5) $\neg \left(E(d) \vee O(d) \right)$
(6) $\forall n \in \mathbb{N} \left(E(n) \vee O(n) \right)$.
(7) $\exists n \in \mathbb{N} \left(E(n) \right)$
(8) $\forall S \subseteq \mathbb{N} \exists n \in S \left(E(n) \right)$.

Exercise 3.18. Can you *simplify* the following mathematical statements? That is, can you find a way to express the same statement using fewer symbols:

(1) $\neg\neg(P \vee Q)$
(2) $\neg(P \vee \neg Q)$
(3) $\neg \forall x \in X, \neg P(x)$
(4) $\neg \exists y \in Y, \neg Q(y)$

4. On implication and proof

Given statements $P \Rightarrow Q$, it is either a true or false statement, and this doesn't depend on the truth of P or Q. As an example let $P(n)$ be "n is divisible by 10" and let $Q(n)$ be "n is even." Then we claim

$$P(n) \Rightarrow Q(n)$$

is a true statement. It's totally fine that P can be false (look at $P(13)$). What matters is the implication.

Question 4.1. Suppose we know P is true and $P \Rightarrow Q$ is true. What can we say about Q?

Answer 4.2. We can deduce that Q must be true!

This type of reasoning is called *deduction* or *logical inference*. We usually write it as

$$\frac{\begin{array}{c}\text{statement}\\ \text{statement}\\ \vdots\\ \text{statement}\end{array}}{\text{conclusion}}$$

The logical rule we just outlined is called *modus ponens*

$$\frac{\begin{array}{c}P \Rightarrow Q\\ P\end{array}}{Q}$$

We have two more important rules of deduction.

Definition 4.3. *Modus tollens* is the deduction

$$\frac{\begin{array}{c}P \Rightarrow Q\\ \neg Q\end{array}}{\neg P}$$

If P implies Q, and Q is false, then P cannot be true.

Definition 4.4. *Elimination* is the deduction

$$\frac{\begin{array}{c}P \vee Q\\ \neg P\end{array}}{Q}$$

If P or Q is true, and P is false, then Q must be true.

Definition 4.5. A *theorem* is a mathematical statement that is true, and a *proof* is a line of deduction that demonstrates its truth from other statements that are known to be true. A *lemma* is a mathematical statement proved on the way to proving a theorem. A *corollary* is a result that follows from the statement of a theorem. A *proposition* is another word for a mathematical statement that is asserted to be true, often a smaller or more obvious result than a lemma or theorem.[13]

How do we prove a mathematical statement? There are lots of different ways to prove something, and it depends upon how the statement is phrased. We'll go through some proof styles and give examples.

4.1. **Direct proof.** Suppose we want to prove a proposition of the following form.

Proposition 4.6. If P then Q.

This doesn't mean that either P or Q are necessarily true, it just means that P will *imply* Q (symbolically, $P \Rightarrow Q$).

A valid line of reasoning here is called *direct proof*. We suppose that P is true, we carry out some logical inference, and we arrive at the conclusion that Q is true.

Proof. Suppose P. Then ... therefore Q. □

Definition 4.7. We say a number $n \in \mathbb{N}$ is *odd* if $n = 2k + 1$ for some $k \in \mathbb{N}$. We say $n \in \mathbb{N}$ is *even* if it is of the form $n = 2k$ for some $k \in \mathbb{Z}$.

[13] The lines between "proposition"/"lemma"/"theorem" are blurry and often very subjective.

Proposition 4.8. If x is odd then x^2 is odd.

We can start by filling out the start and bottom of the proof:

Proof. Suppose x is odd.

...

Therefore x^2 is odd. □

We should *use the definition*.

Proof. Suppose x is odd. Then $x = 2a + 1$ for some integer a, by definition of an odd number.
...
Then $x^2 = 2b + 1$. Therefore x^2 is odd. □

Now we need some line of reasoning to fill in the gaps. Here it comes from expanding x^2 in terms of a, and finding the right expression for b:

Proof. Suppose x is odd. Then $x = 2a + 1$ for some integer a, by definition of an odd number. We have that

$$x^2 = (2a + 1)^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1.$$

Let $b = 2a^2 + 2a$. Then $x^2 = 2b + 1$. Therefore x^2 is odd. □

4.2. **Proof by case.** Suppose I want to prove something is true for all elements in a set X. It might be easier to break X into two smaller sets $X = X_1 \cup X_2$, and do two proofs — prove the statement for elements in X_1 and the statements for elements in X_2.

For example, when proving things about natural numbers $n \in \mathbb{N}$, it can occasionally be helpful to break into two cases: when n is even and when n is odd.

Proposition 4.9. For any $n \in \mathbb{N}$, the number $n^2 + 3n + 1$ is odd.

Proof. First suppose n is even. Then $n = 2k$ for some $k \in \mathbb{N}$. Then we have that

$$n^2 + 3n + 1 = (2k)^2 + 3(2k) + 1 = 6k^2 + 6k + 1 = 2(3k^2 + 3k) + 1.$$

Letting $b = 3k^2 + 3k$, we have that $n^2 + 3n + 1 = 2b + 1$, so $n^2 + 3n + 1$ is odd.

Suppose n is odd. Then $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then we have that

$$\begin{aligned} n^2 + 3n + 1 &= (2k + 1)^2 + 3(2k + 1) + 1 \\ &= (4k^2 + 4k + 1) + (6k + 3) + 1 \\ &= 4k^2 + 10k + 5 \\ &= 2(2k^2 + 5k + 2) + 1. \end{aligned}$$

Letting $b = 2k^2 + 5k + 2$, we have that $n^2 + 3n + 1 = 2b + 1$, hence $n^2 + 3n + 1$ is odd. □

Exercise 4.10. Give a *direct proof* of the following proposition: if x and y are odd, then xy is odd.

Exercise 4.11. Let $x, y \in \mathbb{R}$ be positive real numbers. Give a *direct proof* of the following: if $x < y$ then $x^2 \leq y^2$.

Definition 4.12. Given $a, b \in \mathbb{Z}$, we say a *divides* b, and write $a \mid b$, if $b = ak$ for some $k \in \mathbb{Z}$.

Observe as an example that n is even if and only if $2 \mid n$.

Exercise 4.13. Prove that if $7 \mid 6a$ then $7 \mid a$.

Exercise 4.14. Prove that if $2 \mid 3a$ then $2 \mid a$.

Exercise 4.15. The previous two exercises seem to suggest that, for $a, b, c \in \mathbb{N}$, if $a \mid bc$ then $a \mid b$ or $a \mid c$. Is this always true?

Exercise 4.16. Let $a, b, c \in \mathbb{Z}$. Prove that if $a^2 \mid b$ and $b^3 \mid c$ then $a^6 \mid c$.

Exercise 4.17. If $n \in \mathbb{N}$, prove that $\binom{2n}{n}$ is even.

5. Contrapositive proof

Suppose we want to argue that P implies Q. As we have seen, in order to prove that $P \Rightarrow Q$ directly, we suppose P as a hypothesis, we carry out some logical deductions, and then we arrive at Q. Recall the truth table for $P \Rightarrow Q$:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

We can compare this with the truth table for $(\neg Q) \Rightarrow (\neg P)$:

P	Q	$(\neg Q) \Rightarrow (\neg P)$
T	T	T
T	F	F
F	T	T
F	F	T

They are the same! What this means is that a proof that $P \Rightarrow Q$ is a valid proof that $(\neg Q) \Rightarrow (\neg P)$ and vice versa. This gives us a new style of proof we can try to carry out.

Goal: Prove that $P \Rightarrow Q$.

Strategy 1 (direct proof): Assume P as a hypothesis, carry out some reasoning, and show that you arrive at Q.

Strategy 2 (contrapositive proof): Assume $\neg Q$ as a hypothesis, carry out some reasoning, and show that you arrive at $\neg P$.

Let's see two examples.

Proposition 5.1. Let $x \in \mathbb{Z}$. If $9x + 5$ is even, then x is odd.

Here P is the statement "$9x + 5$ is even" while Q is the statement "x is odd."

Direct proof. Let's prove $P \Rightarrow Q$. Suppose $9x + 5$ is even. Then $9x + 5 = 2k$ for some integer $k \in \mathbb{Z}$. Subtracting $8x$ from each side, we get

$$x = 2k - 8x + 5.$$

Letting $b = k - 4x + 2$, we have that

$$x = 2b + 1,$$

hence x is odd. □

Contrapositive proof. Let's prove $\neg Q \Rightarrow \neg P$. Suppose x is not odd (meaning x is even). Then $x = 2a$ for some $a \in \mathbb{Z}$. Then

$$9x + 5 = 18a + 5 = 2(9a + 2) + 1.$$

Therefore $9x + 5$ is odd. □

Which proof do we prefer? They are both completely valid, but the second one seems a little cleaner. This is because it's easy to take info about x and turn it into info about $9x + 5$, but it's more cumbersome to go the other way around.

Proposition 5.2. Let $x \in \mathbb{Z}$. If $x^2 + 4x + 3$ is even, then x is odd.

Direct proof. Suppose $x^2 + 4x + 3$ is even, so

$$x^2 + 4x + 3 = 2k$$

for some k. Then... bleh. □

Contrapositive. Suppose x is even, then $x = 2n$ for some $n \in \mathbb{Z}$. Then

$$x^2 + 4x + 3 = (2n)^2 + 4(2n) + 3 = 4n^2 + 8n + 3 = 2(2n^2 + 4n + 2) + 1.$$

Letting $c = 2n^2 + 4n + 2$, we have that $x^2 + 4x + 3 = 2c + 1$, so $x^2 + 4x + 3$ is odd. □

Proposition 5.3. Let $x, y \in \mathbb{Z}$. If $3 \mid (xy)$ then $3 \mid x$ or $3 \mid y$.

What is the contrapositive of this statement?

- ▷ P is the statement $3 \mid (xy)$
- ▷ Q is the statement "$3 \mid x$ **or** $3 \mid y$"

The contrapositive is $(\neg Q) \Rightarrow (\neg P)$. The negation of Q is "$3 \nmid x$ *and* $3 \nmid y$."

5.1. **Beware the fallacy of the converse.** Consider the statement $P \Rightarrow Q$. We've defined

- ▷ its *converse*, which is $Q \Rightarrow P$
- ▷ its *contrapositive*, which is $\neg Q \Rightarrow \neg P$.

The statement $P \Rightarrow Q$ is *equivalent* to its contrapositive. That's why we can prove $P \Rightarrow Q$ by proving $(\neg Q) \Rightarrow (\neg P)$. Don't get the contrapositive and converse mixed up though.

Fallacy of the converse: We know $P \Rightarrow Q$ is true. Suppose Q, then we can conclude P.

Exercise 5.4. Write out the contrapositive of the following statements (but don't prove them):

(1) If a is even then $3a$ is even.
(2) If $x > 0$ then $x^2 > 0$
(3) If $x \geq 0$ then $x^2 \geq 0$.
(4) If xy is even then x is even or y is even.
(5) If S is a finite set then every subset of S is finite.
(6) If $A \subseteq B$ then $A \subseteq C$.
(7) If $S \subseteq \mathbb{N}$ and S is nonempty then S contains an even integer or S contains an odd integer.

Exercise 5.5. Mathematicians are sloppy, so you might see any of the following three statements written to mean the same thing:

- ▷ If $S \subseteq \mathbb{N}$ and S is nonempty then S contains an even integer or S contains an odd integer.
- ▷ Let $S \subseteq \mathbb{N}$. If S is nonempty then S contains an even integer or S contains an odd integer.
- ▷ If $S \subseteq \mathbb{N}$ is nonempty, then S contains an even integer or S contains an odd integer.

Discuss: Are these the same statement? Do they have the same contrapositive? Which one is the most clear for you as a reader?[14]

Exercise 5.6. Consider the statement: if $x + y$ is odd then x is odd or y is odd.

(1) Prove this directly (you may need to use cases)
(2) Prove this by contrapositive

Exercise 5.7. (Hammack 4.17) Prove directly or by contrapositive: if n is odd then $8 \mid (n^2 - 1)$.

Exercise 5.8. (Hammack 4.15) Prove directly or by contrapositive: if $x^3 - 1$ is even then x is odd

Exercise 5.9. (Hammack 4.28) Prove directly or by contrapositive: if $n \in \mathbb{Z}$ then $4 \nmid (n^2 - 3)$.

[14] Further exercise – write proofs very clearly! In sentences that look identical, like those above, there's a lot of room for logical ambiguity.

6. CONTRADICTION, IF AND ONLY IF

Suppose we want to prove a direct statement $P \Rightarrow Q$. We can look at the truth table, and we see there's only one way that $P \Rightarrow Q$ could fail to hold, namely if P is true and Q is false:

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

So let's show this possibility *can never occur*. How would we do that without a direct proof?

Suppose someone walks up to you and says "I have this mathematical proposition P, and if P is true, then it logically follows that 2 is odd." You might say, "good for you, but whatever P is, it can't be true because 2 isn't odd, so I'm going to conclude that P is false." That's the point of contradiction! We're going to show something is false by using it to arrive at some kind of contradiction.

Definition 6.1. To prove $P \Rightarrow Q$ *by contradiction*, we assume P *and* we assume $\neg Q$, and then we carry out some formal deduction to arrive at a contradiction, i..e we show some other statement R and its converse $\neg R$ are both true.

Proposition 6.2. If $x + y$ is odd, then x is odd or y is odd.

Proof by contradiction. Suppose $x + y$ is odd, and suppose towards a contradiction that both x and y are even. Then $x + y$ is even, contradicting that $x + y$ is odd. □

We assumed P and $\neg Q$, and we arrived at $\neg P$. Since both P and $\neg P$ can't be true, we must have that $P \wedge \neg Q$ is false. Hence $P \Rightarrow Q$ is true.

Definition 6.3. A natural number $n \geq 2$ is *prime* if it is its only divisors are 1 and itself.

Remark 6.4. The number 1 isn't prime by convention.

Every integer factors *uniquely* as a product of primes:

$$12 = 2^2 \cdot 3$$
$$51840 = 2^7 \cdot 3^4 \cdot 5.$$

So just as all molecules are built of atoms, all integers are built out of prime numbers. They are the *building blocks* of numbers.

Lemma 6.5. Suppose p is a prime number and $n \in \mathbb{N}$. Then we cannot have both $p \mid n$ and $p \mid n + 1$.

Theorem 6.6 (Euclid, 300BC)**.** There are infinitely many prime numbers.

Proof. Suppose towards a contradiction there were only finitely many. Write them out as $p_1, \ldots, p_k$. Let

$$N = p_1 \cdot p_2 \cdots p_k + 1.$$

Then by the lemma, N is not divisible by any of the primes $p_1, \ldots, p_k$. However every number decomposes uniquely into primes — this means that N is divisible by some prime *not on our list.* This contradicts that $p_1, \ldots, p_k$ were the only prime numbers. □

Definition 6.7. A number is *rational* if it is of the form $\frac{a}{b}$ for $a, b \in \mathbb{Z}$ and $b \neq 0$. A number is *irrational* if it is not of this form.

Observe that we can always write a rational number in *reduced* form, which means a and b have no common multiples. For instance $\frac{16}{12}$ isn't reduced, since the top and bottom share a factor of 4, but we can write it in a reduced form as $\frac{4}{3}$.

Lemma 6.8. (Exercise on the homework) Let $a \in \mathbb{Z}$. If a^2 is even, then a is even.

Theorem 6.9. The quantity $\sqrt{2}$ is not rational.

Proof. Suppose towards a contradiction that $\sqrt{2}$ was rational. Then we can write $\sqrt{2} = \frac{a}{b}$, and we can assume this fraction is reduced (so that a and b have no common factors). Then

$$2 = \frac{a^2}{b^2},$$

so we have that

$$2b^2 = a^2.$$

This means that a^2 is even. By the lemma this implies a is even, so we can write $a = 2k$ for some $k \in \mathbb{Z}$. Let's expand the equation above:

$$2b^2 = a^2 = (2k)^2 = 4k^2.$$

We can divide by a 2 on both sides, and we get

$$b^2 = 2k^2.$$

This implies b^2 is even, which by the lemma implies b is even. So a and b are both even, meaning they are both divisible by 2. This contradicts our assumption that a/b was a reduced fraction. □

Exercise 6.10. Suppose $n \in \mathbb{Z}$. We want to show that if n is odd then n^2 is odd.

(1) Think about how a direct proof, a proof by contrapositive, and a proof by contradiction would go
(2) Which proof technique feels the most natural here?

Exercise 6.11. Prove that $\sqrt{3}$ is irrational.

Exercise 6.12. Prove that $\sqrt[3]{2}$ is irrational.

Exercise 6.13. Prove that there exist no integers a and b for which $21a + 30b = 1$.

Exercise 6.14. Prove there is no largest integer. Prove there is no smallest positive rational number.

Exercise 6.15. (Bonus – some more about truth tables);

(1) Try to create an expression with P and Q that leads to the following truth table:

P	Q	your expression
T	T	F
T	F	T
F	T	T
F	F	F

You're allowed to use $\neg$, $\vee$, $\wedge$, etc.

(2) Use your expression above and another expression to get the following outputs:

P	Q	expression 1	expression 2
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	F

(3) Turn all the T's into 1's and all the F's into 0's. What did these expressions do? Ask for a hint if you get stuck.
(4) Try to do the next step up (four inputs, four outputs).

7. Modular arithmetic

When we write $22/7 = 3\frac{1}{7}$, we are really using the fact that

$$22 = 3 \cdot 7 + 1.$$

Here this 1 is the *remainder* when we divide 22 by 7. We're going to state a theorem that shows such an expression is always unique.

Theorem 7.1 (Euclidean Division Algorithm)**.** Let n and $d \geq 1$ be integers. Then there exist uniquely determined integers q and r so that

$$n = qd + r$$

for $0 \leq r < d$.

Proof. We are given n and d and want to find q and r. In particular we want r to be as small as possible, so we want to minimize the value $n - qd$, while still keeping it non-negative. To that end, let's take the set

$$X = \{n - td \colon t \in \mathbb{Z},\ n - td \geq 0\}.$$

We first claim X is nonempty. This is true because if $n \geq 0$, then $n = n - 0d$ is in X, and if $n < 0$ then $n - nd = n(1 - d)$ is in X.

Since X is nonempty we can let r be the smallest member of X. Then $r = n - qd$ for some $q \in \mathbb{Z}$. We still have to show that

(1) $0 \leq r < d$, and
(2) r and q are *uniquely determined*.

For the first step, suppose towards a contradiction that $r \geq d$. Then we can write

$$0 \leq r - d = n - (q + 1)d.$$

Hence $r - d$ is in X, but $r - d < r$, contradicting the minimality of r. Hence we conclude that $0 \leq r < d$.

To show uniqueness, suppose that $n = q'd + r'$ with $0 \leq r' < d$. Let's first assume that $r \leq r'$. Then we have that

$$(q - q')d = r' - r \leq r' < d$$

so $r' - r$ is a nonnegative multiple of d which is less than d, which can only happen if $r' - r = 0$. Hence $r' = r$ and $q = q'$. The case where $r \geq r'$ can be done similarly. □

Definition 7.2. We say two integers a and b are *congruent modulo* n if $n \mid (a - b)$. In this case we write

$$a \equiv b \pmod{n}.$$

Corollary 7.3. Every integer a admits a unique *remainder* modulo n, which we denote by $\bar{a}$, for which $0 \leq \bar{a} < n$.

Proof. By the division algorithm, we have a unique expression

$$a = qn + \bar{a}$$

for some $0 \leq \bar{a} < n$. □

Let's define a new set

$$\mathbb{Z}/n\mathbb{Z} := \left\{\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}\right\}.$$

We can add and multiply things in this set, and this is called *modular arithmetic*.[15]

Example 7.4. Working with $\mathbb{Z}/12$ is essentially what we do when we tell time – $\bar{9} + \bar{4} = \overline{13} = \bar{1}$ in $\mathbb{Z}/12$, since four hours past nine, it will be one o'clock.

We'll get into modular arithmetic a lot more on the worksheet and after the midterm when we study group theory.

Exercise 7.5. Determine if the following are true or false:

(1) $40 \equiv 13 \pmod 9$
(2) $-29 \equiv 1 \pmod 7$
(3) $-29 \equiv 6 \pmod 7$
(4) $132 \equiv 0 \pmod{11}$.

Exercise 7.6. For which $k \in \mathbb{Z}$ are the following true:

(1) $4 \equiv 2k \pmod 7$
(2) $12 \equiv 3k \pmod{10}$
(3) $3k \equiv k \pmod 9$
(4) There exists some $b \in \mathbb{Z}$ for which $kb \equiv 1 \pmod 5$.
(5) There exists some $b \in \mathbb{Z}$ for which $kb \equiv 1 \pmod 6$.

Exercise 7.7. Prove that

$$1 + 2 + 3 + 4 + \ldots + n = \frac{n^2 + n}{2}$$

for every $n \geq 1$.

Exercise 7.8. For each $n \in \mathbb{N}$, prove that

$$2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2.$$

Exercise 7.9. Prove that $n^3 - n \equiv 0 \pmod 3$ for each $n \geq 1$.

Exercise 7.10. Prove that $3^{3n} + 1$ is a multiple of 7 for all odd $n \geq 0$.

Exercise 7.11. For this exercise let's work modulo 2, so there are two operations, corresponding to even and odd.

(1) Fill out the table for *addition* modulo two:

p	q	$p+q$
$\bar{1}$	$\bar{1}$	
$\bar{1}$	$\bar{0}$	
$\bar{0}$	$\bar{1}$	
$\bar{0}$	$\bar{0}$	

(2) Fill out the table for *multiplication* modulo two:

p	q	$p \cdot q$
$\bar{1}$	$\bar{1}$	
$\bar{1}$	$\bar{0}$	
$\bar{0}$	$\bar{1}$	
$\bar{0}$	$\bar{0}$	

(3) Do these tables look familiar...? Discuss!

[15] Really what we're doing is taking the set of all $\bar{k}$ for $k \in \mathbb{Z}$, and then saying $\bar{k} = \overline{k+n} = \overline{k+2n} = \cdots$. This comes from something called an *equivalence relation* that we won't get to in this class.

8. Mathematical induction

Suppose we're bored in class and we start adding odd numbers starting at one:

$$1 = 1$$
$$1 + 3 = 4 = 2^2$$
$$1 + 3 + 5 = 9 = 3^2$$
$$1 + 3 + 5 + 7 = 16 = 4^2.$$

We start to see a pattern, and we might want to guess that it is always true:

Conjecture 8.1. The sum of the first n odd natural numbers equals n^2. In other words,

$$1 + 3 + \ldots + (2n - 1) = n^2.$$

It's not obvious how to prove something like this – we will discuss a *new proof technique*.

Induction: Let $P(n)$ be a mathematical statement about $n \in \mathbb{N}$ that we want to prove for all $n \in \mathbb{N}$. In the example above, $P(n)$ might be the statement "the sum of the first n odd natural numbers equals n^2."

Suppose we can prove that

(1) $P(1)$ is true (the *base case*)
(2) $P(n)$ being true implies that $P(n + 1)$ is true (the *inductive step*)

Then we claim we know that $P(n)$ is true for all $n \in \mathbb{N}$! Why is this true? Because we know $P(1)$ is true, and since $P(1)$ implies $P(2)$ we know that $P(2)$ is true as well, and since $P(2)$ implies $P(3)$, we know that $P(3)$ is true as well, and so on...

We think about this like dominoes falling.

So let's prove the conjecture!

Proof. **Base case**: If $n = 1$, then it is clear that $1 = 1^2$ so we are done.[16]

Inductive step: Suppose we know that

$$1 + 3 + \cdots + (2n - 1) = n^2.$$

We want to argue that this implies that $1 + 3 + \ldots + (2n - 1) + (2n + 1)$ is equal to $(n + 1)^2$. Let's expand:

$$(n + 1)^2 = n^2 + 2n + 1.$$

We can *apply the inductive hypothesis* to plug in the value for n^2, and we get

$$\begin{aligned}(n + 1)^2 &= n^2 + 2n + 1 \\ &= (1 + 3 + \ldots + (2n - 1)) + (2n + 1).\end{aligned}$$

Hence $(n + 1)^2$ is equal to the sum of the first $n + 1$ odd numbers, and we are done! □

Proposition 8.2. For every $n \geq 4$, we have that $2^n < n!$.

Proof. **Base case**: If $n = 4$, then $2^4 = 16 < 24 = 4!$.

Inductive step: Suppose $2^n < n!$. Then

$$2^{n+1} = 22^n < 2n! < (n + 1)n! = (n + 1)!,$$

since $2 < n + 1$ for $n \geq 2$ (in particular for $n \geq 4$). □

Strong induction: In trying to prove $P(n)$ implies $P(n+1)$ we might sometimes want to assume not just that $P(n)$ holds but that $P(k)$ holds for all $k \leq n$. This is also a valid proof technique called *strong induction*

[16] Base cases can often be the easiest part of a proof by induction, but they are necessary to include!

(1) $P(1)$ is true (base case)
(2) $P(1) \wedge P(2) \wedge \cdots \wedge P(n)$ implies $P(n+1)$ (strong inductive step).

Proposition 8.3. Every integer $n \geq 2$ is a product of primes.

Proof. **Base case**: $n = 2$ is itself a prime.

Inductive step. Suppose we know every integer $k \leq n$ is a product of primes. We want to show $n+1$ is a product of primes. If $n+1$ is itself a prime, then we are done. If not, then it factors as

$$n + 1 = ab,$$

where $2 \leq a, b < n$. Then each of a and b factor as a product of primes by the inductive hypothesis, hence $n+1$ is a product of all the primes in a and in b. □

Proposition 8.4. We have that $5^{2n} \equiv 1 \pmod{24}$ for each $n \geq 0$.

Proof. We prove by strong induction. Suppose we know that $5^{2k} \equiv 1 \pmod{24}$ for each $k \leq n$. We want to prove it for $n+1$. We can write

$$5^{2(n+1)} - 1 = 5^{2n+2} - 1 = (5^{n+1} - 1)(5^{n+1} + 1).$$

By strong induction, $5^{n+1} \equiv 1 \pmod{24}$, hence $24 \mid (5^{n+1} - 1)$, and hence 24 divides the entire product above. □

Exercise 8.5. Determine if the following are true or false:

(1) $40 \equiv 13 \pmod{9}$
(2) $-29 \equiv 1 \pmod{7}$
(3) $-29 \equiv 6 \pmod{7}$
(4) $132 \equiv 0 \pmod{11}$.

Exercise 8.6. For which $k \in \mathbb{Z}$ are the following true:

(1) $4 \equiv 2k \pmod{7}$
(2) $12 \equiv 3k \pmod{10}$
(3) $3k \equiv k \pmod{9}$
(4) There exists some $b \in \mathbb{Z}$ for which $kb \equiv 1 \pmod{5}$.
(5) There exists some $b \in \mathbb{Z}$ for which $kb \equiv 1 \pmod{6}$.

Exercise 8.7. Prove that

$$1 + 2 + 3 + 4 + \ldots + n = \frac{n^2 + n}{2}$$

for every $n \geq 1$.

Exercise 8.8. For each $n \in \mathbb{N}$, prove that

$$2^1 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2.$$

Exercise 8.9. Prove that $n^3 - n \equiv 0 \pmod{3}$ for each $n \geq 1$.

Exercise 8.10. Prove that $3^{3n} + 1$ is a multiple of 7 for all odd $n \geq 0$.

Exercise 8.11. For this exercise let's work modulo 2, so there are two operations, corresponding to even and odd.

(1) Fill out the table for *addition* modulo two:

p	q	$p+q$
$\bar{1}$	$\bar{1}$	
$\bar{1}$	$\bar{0}$	
$\bar{0}$	$\bar{1}$	
$\bar{0}$	$\bar{0}$	

(2) Fill out the table for *multiplication* modulo two:

p	q	$p \cdot q$
$\bar{1}$	$\bar{1}$	
$\bar{1}$	$\bar{0}$	
$\bar{0}$	$\bar{1}$	
$\bar{0}$	$\bar{0}$	

(3) Do these tables look familiar...? Discuss!

9. FUNCTIONS

Given two sets X and Y, a *function* from X to Y is a way to take as input elements in X and output elements in Y. We write $f\colon X \to Y$ to say "f is a function from X to Y."

Notation 9.1. We write $f(x) = y$ to mean that, for the function $f\colon X \to Y$, x is sent to y. We might also write $x \mapsto y$. This arrow $\mapsto$ is
`mapsto` in LaTeX, and means x gets "mapped to" some value by a function.

[draw some functions, see which are valid]
Q: What are some properties that functions can have?

Definition 9.2. A function $f\colon X \to Y$ is *surjective* or *onto* if every element $y \in Y$ is mapped to by some $x \in X$. More explicitly, for every $y \in Y$ there exists some $x \in X$ (maybe more than one) so that $f(x) = y$.

Question 9.3. If X and Y are finite, and $f\colon X \to Y$ is some surjective function, what can we say about the *sizes* of X and Y?

Definition 9.4. A function $f\colon X \to Y$ is *injective* or *one-to-one* if no two elements in X map to the same element in Y. In other words if $a, b \in X$ and $a \neq b$ then $f(a) \neq f(b)$.

Giving a direct proof of injectivity using this definition is a little hard when X is a very big set. Easier is to prove the contrapositive! That is, you suppose $f(a) = f(b)$ for some $a, b \in X$ and then argue that $a = b$.

Question 9.5. If X and Y are finite, and $f\colon X \to Y$ is some *injective* function, what can we say about the *sizes* of X and Y?

Definition 9.6. We say a function $f\colon X \to Y$ is *bijective* if it is both injective and surjective.

Question 9.7. If X and Y are finite, and $f\colon X \to Y$ is some *bijective* function, what can we say about the *sizes* of X and Y?

The following example is goofy?

Proposition 9.8. (From Hammack, p.234) There are two people in the state of Texas with the same number of hairs on their head.

Proof. There exists some function

$$\{\text{people in Texas}\} \to \mathbb{N},$$

counting the number of hairs on every Texan's head. Biology tells us every human has less than 1 million hairs on their head, so let's actually revise this function to write

$$\{\text{people in Texas}\} \to \{1, 2, 3, 4, \ldots, 1 \text{ million}\}.$$

And the population of Texas is around 31.29 million (in 2024), so the size of the set on the left is much bigger than the size of the set on the right! If this function were injective, then because both sets are finite, the size of the set on the left would have to be smaller than the size of the set on the right. Hence this function is not injective! Meaning there are two people in Texas with the same number of hairs on their head. □

This is what's known as the *pigeonhole principle*

Pigeonhole principle: If X and Y are finite sets and $f\colon X \to Y$ is any function, then

(1) If $|X| > |Y|$ then f is not injective
(2) If $|X| < |Y|$ then f is not surjective.

Example 9.9. Most things you've seen in calculus are *functions* from $\mathbb{R} \to \mathbb{R}$. We write $f(x) = \cos(x)$ for instance to refer to the function $\mathbb{R} \to \mathbb{R}$ sending $x \mapsto \cos(x)$.

Example 9.10. The function

$$\begin{aligned} f\colon \mathbb{R} \smallsetminus \{0\} &\to \mathbb{R} \\ x &\mapsto \frac{1}{x} + 1 \end{aligned}$$

is injective but not surjective.

Proof. Suppose we have $a, b \in \mathbb{R} \smallsetminus \{0\}$ so that $f(a) = f(b)$. That is,

$$\begin{aligned} \frac{1}{a} + 1 &= \frac{1}{b} + 1 \\ \frac{1}{a} &= \frac{1}{b}. \end{aligned}$$

Inverting we get $a = b$. It is not surjective because 1 is not hit. □

9.1. **Bijectivity and size.** We have seen at least for finite sets that the existence of a bijection $X \to Y$ means that X and Y have the same size. To that end, bijections are a great way to talk about the *size* of two sets. If you can prove that no bijection exists between X and Y then they must have different sizes.

What about infinite sets?

Definition 9.11. We say a set X is *countable* or *countably infinite* if there exists a bijection $f\colon \mathbb{N} \to X$.

We say this because this means all the elements of X can be labeled as $f(0), f(1), f(2), f(3)$, etc. In other words they can be *counted*.

Question 9.12. Does there exist a bijection $\mathbb{N} \to [0, 1]$? Certainly there exist many injections, one of which is given by sending $n \mapsto 1/n$ and $0 \mapsto 0$. This means that $|\mathbb{N}| \leq |[0, 1]|$.

Theorem 9.13 (Cantor diagonalization)**.** There exists *no bijection*

$$\mathbb{N} \to [0, 1].$$

Proof. Suppose towards a contradiction that there did exist some bijection, call it f. Then we will derive a contradiction by explicitly constructing an element $w \in [0, 1]$ which is *not of the form* $f(n)$ for any $n \geq 0$. Here's how we do this – we write out the decimal expansion of each value of f.

$$\begin{aligned} f(0) &= 0.373294170300213890021... \\ f(1) &= 0.721092380414890237298... \\ f(2) &= 0.139174912828949921322... \\ f(3) &= 0.775999030321390321321... \\ f(4) &= 0.389020391888329923021... \\ &\vdots \end{aligned}$$

We then go through and build a new number w whose decimal expansion is different than $f(n)$ at level n.

$$\begin{aligned} f(0) &= 0.\mathbf{3}73294170300213890021... \\ f(1) &= 0.7\mathbf{2}109238041489023729 8... \\ f(2) &= 0.13\mathbf{9}1749128289499213 22... \\ f(3) &= 0.772\mathbf{5}99903032139032132... \\ f(4) &= 0.3891\mathbf{0}20391888329923021... \\ &\dots \\ w &= 0.\mathbf{43061}\dots \end{aligned}$$

□

What does this tell us? $\mathbb{N}$ and $[0,1]$ are clearly both infinite sets, but one of them is *strictly smaller* than the other. We might say $[0,1]$ is *uncountably infinite*. We have proven that there exist different sizes of infinity!

Exercise 9.14. Prove that $f\colon \mathbb{Q} \to \mathbb{Q}$, defined by $x \mapsto 2x$, is a bijection.

Exercise 9.15. Prove that $f\colon \mathbb{Z} \to \mathbb{Z}$, defined by $x \mapsto 2x$, is not a bijection.

Definition 9.16. Given a function $f\colon X \to Y$, we define its *graph* Γ_f to be the following subset:[17]

$$\Gamma_f := \{(x, f(x))\colon x \in X\} \subseteq X \times Y.$$

Exercise 9.17. Convince yourself that this definition gels with the definition of a "graph of a function" that you learned in high school algebra.

But not every subset of $X \times Y$ is the graph of a function!

Exercise 9.18. Suppose you have some subset $A \subseteq X \times Y$. What do you need to know about A in order to say that A is the graph of some function $X \to Y$?

Exercise 9.19. Let's write $X \cong Y$ as shorthand for "there exists a bijection $X \to Y$"[18] Prove that this relation is

(1) *reflexive* – meaning $X \cong X$ for every set X
(2) *symmetric* – if $X \cong Y$ then $Y \cong X$
(3) *transitive* – if $X \cong Y$ and $Y \cong Z$ then $X \cong Z$

Exercise 9.20. Which of the following sets are in bijection with one another?

(1) the natural numbers $\mathbb{N}$
(2) the integers $\mathbb{Z}$
(3) the rational numbers $\mathbb{Q}$
(4) the real numbers $\mathbb{R}$
(5) the open interval $(0,1)$
(6) the open interval $(0,2)$

A subset $A \subseteq X$ is said to be *proper* if $A \neq X$. In other words there exists some $x \in X$ so that $x \notin A$. We sometimes write $A \subsetneq X$ to notationally clarify that a subset is intended to be proper.

Exercise 9.21. What kinds of sets are bijective to proper subsets of themselves?

[17] The notation Γ_f is pretty standard – here Γ is capital "gamma" and we're supposed to think of this "g" as standing for "graph."

[18] If you use this outside this math class, other mathematicians might ask you to justify the notation. In this case you're allowed to say the special phrase "they are isomorphic in the category of sets" in which case the other mathematician will begrudgingly approve.

10. PERMUTATIONS

A *permutation* is a way to rearrange sets.

Definition 10.1. A *permutation* of a set X is any bijection from a set X to itself.

Example 10.2. Let $n = 3$, then we can study all the permutations from $\{1, 2, 3\}$ to itself:

$1 \to 1$	$1 \to 1$	$1 \to 2$	$1 \to 2$	$1 \to 3$	$1 \to 3$
$2 \to 2$	$2 \to 3$	$2 \to 1$	$2 \to 3$	$2 \to 1$	$2 \to 2$
$3 \to 3$	$3 \to 2$	$3 \to 3$	$3 \to 1$	$3 \to 2$	$3 \to 1$

Notation 10.3. We denote by Σ_n the set of permutations of $\{1, 2, 3, \ldots, n\}$. In other words

$$\Sigma_n := \{f \colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\} : f \text{ is bijective}\}.$$

What's a concise way to write a permutation? We can write the numbers $\{1, \ldots, n\}$ across the top row, and where each of them is mapped to along the bottom row. Then our 6 permutations above can all be written as

$$\Sigma_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right\}.$$

Definition 10.4. If $\sigma, \tau \in \Sigma_n$, we define their *multiplication* $\sigma \cdot \tau$ to be the permutation sending $k \in \{1, \ldots, n\}$ to $\sigma(\tau(k))$.

Example 10.5. If $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$, then

$$\begin{aligned} (\sigma \cdot \tau)(1) &= \sigma(\tau(1)) = \sigma(2) = 1 \\ (\sigma \cdot \tau)(2) &= \sigma(\tau(2)) = \sigma(1) = 3 \\ (\sigma \cdot \tau)(3) &= \sigma(\tau(3)) = \sigma(3) = 2. \end{aligned}$$

So we have that

$$\sigma \cdot \tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

We could also see this by stacking τ on top of σ!

Let $e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$. This is called the *identity permutation* because it doesn't do anything.

Note that $e \cdot \sigma = e$ and $\sigma \cdot e = \sigma$. This should remind us of multiplying by 1.

We say another element τ in Σ_n is the *inverse* of σ if $\tau\sigma = e$ and $\sigma\tau = e$.

10.1. Cycles.

Consider

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 4 & 3 & 2 \end{pmatrix}.$$

What does σ do if we repeat it over and over? It cycles $2 \mapsto 5 \mapsto 3 \mapsto 6 \mapsto 2 \mapsto \cdots$, and 1 and 4 stay fixed.

For these kinds of permutations, we have some more concise notation.

Definition 10.6. If $\{k_1, \ldots, k_r\} \subseteq \{1, \ldots, n\}$, we write

$$\sigma = (k_1 \ \ldots \ k_r)$$

for the permutation in Σ_n satisfying

- $\sigma(k_i) = k_{i+1}$ if $1 \leq i < r$
- $\sigma(k_r) = k_1$
- $\sigma(k) = k$ if $k \notin \{k_1, \ldots, k_r\}$

So in *cycle notation*, we have that

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 6 & 4 & 3 & 2 \end{pmatrix} = (2\ 5\ 3\ 6).$$

It doesn't matter where we start the notation, we have that

$$(2\ 5\ 3\ 6) = (5\ 3\ 6\ 2) = (3\ 6\ 5\ 2) = (6\ 5\ 2\ 3) \in \Sigma_6.$$

We would say σ is a 4-*cycle*.

Remark 10.7. Not every permutation is a cycle!

Theorem 10.8. If $\sigma \in \Sigma_n$ is an r-cycle, then σ^{-1} is an r-cycle.

Proof sketch. We want to argue that

$$(k_1\ k_2\ \cdots\ k_r)^{-1} = (k_r\ k_{r-1}\ \cdots\ k_2\ k_1).$$

So we should just verify that

$$\epsilon = (k_1\ k_2\ \cdots\ k_r)(k_r\ k_{r-1}\ \cdots\ k_2\ k_1)$$

and

$$\epsilon = (k_r\ k_{r-1}\ \cdots\ k_2\ k_1)(k_1\ k_2\ \cdots\ k_r).$$

□

What about the following permutation?

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 1 & 6 & 7 & 5 \end{pmatrix}.$$

It's not a single cycle, but the *product of two cycles*. In other words, it can be thought of as the *product* of the two cycle permutations

$$(1\ 2\ 3\ 4) \cdot (5\ 6\ 7).$$

10.2. **Cycle composition.** What happens when we take a product of cycles? We now bump into some annoying notation, for instance consider:

$$(1\ 3\ 2\ 5) \cdot (5\ 7\ 4) \in \Sigma_7.$$

We have $\sigma = (1\ 3\ 2\ 5)$ and $\tau = (5\ 7\ 4)$, so the induced permutation will be

$$\sigma \cdot \tau = (1\ 3\ 2\ 5\ 7\ 4).$$

Definition 10.9. We say two cycles $(k_1\ \cdots k_r)$ and $(m_1\ \cdots\ m_s)$ in Σ_n are *disjoint* if $k_i \neq m_j$ for every $1 \leq i, j \leq n$.

Exercise 10.10. If σ and τ are disjoint cycles then $\sigma\tau = \tau\sigma$.

Definition 10.11. A *cycle decomposition* of a permutation $\sigma \in X_n$ is an expression for it as a product of disjoint cycles.

We should think of *cycles* as analogous to primes in some sense – just as every number factors uniquely into a product of primes, we want to see that every permutation factors uniquely as a product of disjoint cycles.

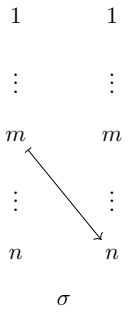
Theorem 10.12 (Cycle decomposition theorem)**.** If $\sigma \in \Sigma_n$ is a non-identity permutation, then σ is a product of (one or more) disjoint cycles of length at least two. This factorization is unique up to the order in which the cycles are multiplied.

We'll prove first *existence* (that every permutation *can* be written as a product of disjoint cycles), then *uniqueness* (that this expression is unique up to reordering the cycles).

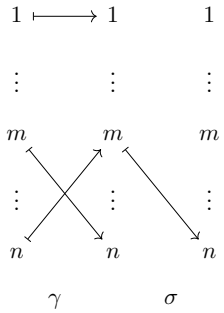
Proof of existence. We will prove this by induction on n. If $n = 2$, then there are only two elements in Σ_2, namely the identity and (1 2). Note that (1 2) is already a cycle, hence we are done.

We proceed by strong induction. Suppose we've proven it for Σ_{n-1}, and take $\sigma \in \Sigma_n$. If $\sigma(n) = n$, then σ just permutes the elements $\{1, \ldots, n-1\}$, in other words it can be thought of as an element in Σ_{n-1}. By induction, we have a disjoint cycle decomposition for σ.

The other case is that σ does not fix n. Since σ is a bijection, it sends *something* to n, and we've assumed this something isn't n itself, so there is some $1 \leq m < n$ so that $\sigma(m) = n$. Here's a cartoon of σ:



Let's define a transposition $\gamma = (m\ n)$ and let τ be the composite $\tau = \sigma\gamma$. Notice what we get



Then $\tau(n) = \sigma(\gamma(n)) = \sigma(m) = n$. Hence $\tau \in \Sigma_{n-1}$, so it admits a decomposition into a product of disjoint transpositions. Let's write it as

$$\tau = c_1 c_2 \cdots c_s,$$

where each of the c_i's is some cycle in Σ_{n-1}, and they are disjoint from one another. Since $\gamma = \gamma^{-1}$, we can write σ in terms of τ and hence in terms of the c_i's:

$$\sigma = \tau\gamma = c_1 c_2 \cdots c_s \gamma.$$

We have two cases to consider.

Case 1: If γ is disjoint from the other c_i's, then we are done![19]

Case 2: We have that γ is not disjoint from the other c_i's. Note that $\tau(n) = n$, so n doesn't appear in any of the c_i's. Hence γ note being disjoint means there is some c_i which moves m. Since we are allowed to reorder, we can assume without loss of generality that it is the last one c_s. We can write

$$c_s = (k_1\ k_2 \cdots k_r\ m)$$

for some $r \geq 1$ and some k_j's. Then we notice that

$$\begin{aligned}\sigma &= c_1 \cdots c_s \gamma \\ &= c_1 \cdots c_{s-1}(k_1\ k_2\ \cdots\ k_r\ m)(m\ n) \\ &= c_1 \cdots c_{s_1}(k_1\ k_2\ \cdots k_r\ m\ n).\end{aligned}$$

Since c_s was disjoint from the other c_i's and none of the other c_i's moved m or n, this is a disjoint cycle decomposition for σ. Hence we're done. □

Proof of uniqueness. This is on the homework! □

Exercise 10.13. Fill out the multiplication table for Σ_3 – do *row* times *column*! I filled out the example from class so that you can check you're doing it right

	$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$	$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$	$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$
$\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}$						
$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$						
$\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}$						
$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$						
$\begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$			$\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}$			
$\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$						

Exercise 10.14. Compute the inverse of

$$\begin{pmatrix}1&2&3&4&5&6&7&8\\5&3&8&2&1&4&7&6\end{pmatrix} \in \Sigma_8.$$

Exercise 10.15. Factor each of the following into disjoint cycles:

(1) $\begin{pmatrix}1&2&3&4&5&6&7&8&9\\4&7&9&8&2&1&6&3&5\end{pmatrix}$

(2) $\begin{pmatrix}1&2&3&4&5&6&7&8&9\\3&8&9&5&2&1&6&4&7\end{pmatrix}$

(3) $\begin{pmatrix}1&2&3&4&5&6&7&8&9\\2&8&6&9&4&7&3&1&5\end{pmatrix}$

(4) $\begin{pmatrix}1&2&3&4&5&6&7&8&9\\6&4&8&9&3&1&7&5&2\end{pmatrix}$

[19] As a side note, observe that γ being disjoint from the c_i's is equivalent to the statement that $\sigma(n) = m$. If we used strong induction we could also have eliminated this case by noting that if $\sigma(m) = n$ and $\sigma(n) = m$, we can view σ as $(m\ n)$ combined with a permutation of the set $\{1, \ldots, n\} \smallsetminus \{m, n\}$ of size $n - 2$, which therefore has a disjoint cycle decomposition.

Exercise 10.16. Write the cycle decompositions of the following products of cycles:

(1) $(1\ 3)(2\ 5\ 7)(3\ 8\ 5)$
(2) $(1\ 2\ 3\ 4\ 5)(6\ 7)(1\ 3\ 5\ 7)(1\ 6\ 3)$

Exercise 10.17. Prove by induction that $|\Sigma_n| = n!$

11. Monoids and groups

We've seen that the set Σ_n of bijections $\{1, \dots, n\}$ can be given a *multiplication*, defined by composing permutations. This has an identity element, has inverses, etc.

We'd like to abstract this a bit.

Definition 11.1. A *binary operation* $\star$ on a set X is a function

$$X \times X \to X.$$

We denote by $\star(x_1, x_2)$ as $x_1 \star x_2$.

Example 11.2. Addition and multiplication are binary operations on $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, etc.

Definition 11.3. We say a binary operation $\star$ on a set X is *associative* if

$$(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3)$$

for every $x_1, x_2, x_3 \in X$.

Every binary operation we will deal with in this class will be associative, but there exist interesting operations which aren't! (look up the *octonions* if you want a particular example).

Definition 11.4. If $(X, \star)$ is a set with a binary operation, we say it has a *unit*, or it is *unital* if there exists an element $e \in X$ for which

$$e \star x = x = x \star e$$

for every $x \in X$.

Theorem 11.5. If $(X, \star)$ is unital, then that unit is unique.

Proof. Suppose $e_1, e_2 \in X$ are both units for $\star$. Then

$$e_1 = e_1 \star e_2 = e_2,$$

by unitality. □

Definition 11.6. We say a set with a binary operation $(X, \star)$ is a *monoid* if it is associative and unital.

Definition 11.7. We say a monoid $(X, \star)$ is *commutative* if $x \star y = y \star x$ for every $x, y \in X$.

Example 11.8. The set of natural numbers with $(\mathbb{N}, +)$ is a commutative monoid[20] with binary operation given by addition

$$\mathbb{N} \times \mathbb{N} \to \mathbb{N}$$
$$(a, b) \mapsto a + b.$$

Proof. We first verify addition is well-defined, this is true since the addition of any two natural numbers is also a natural number. We also check that addition is associative, which is true. Finally, we see $0 + n = n = n + 0$ for any $n \in \mathbb{N}$, and that $x + y = y + x$, so we have a commutative monoid. □

Example 11.9. The set Σ_n of permutations is a monoid (not necessarily commutative)!

[20] This is kind of *why* we want to include 0 as a natural number.

Proof. We verify the composition of two permutations is indeed a permutation, so the binary operation of permutation composition is well-defined. Permutation composition is associative, as we can check, and the identity permutation is an identity element in the monoid sense. □

Example 11.10. If X is a set, then $\mathcal{P}(X)$ is a monoid under union.

Example 11.11. If X is a set, then $\mathcal{P}(X)$ is a monoid under intersection.

Example 11.12. We have that $(\mathbb{Z}, \cdot)$ is a monoid under multiplication. The unit is 1.

Example 11.13. For the CS people – come up with monoid operations on certain data structures!

If we have a binary operation on a set, we can *check* whether it is a monoid or not. In this sense the binary operation is *structure*, whereas being a monoid is a *property* that is either satisfied or failed by that structure.

Question: How many binary operations exist on a finite set X?

If X has n elements, then we are asking how many functions

$$X \times X \to X$$

exist. Note that $X \times X$ has n^2 elements, and for each of these we have n choices of where it can be sent in X. So altogether there are exactly n^{n^2} binary operations on a set X. Let's look at how fast these grow

n	1	2	3	4
n^{n^2}	1	16	19,683	over 4 billion

Question: How many of these are monoids?

Answer: Wide open.

This question is a little too hard. It turns out if we impose one more condition, this type of problem becomes more approachable.

Definition 11.14. Let $(X, \star, e)$ be a monoid, and let $x \in X$ be an arbitrary element. We say $y \in X$ is an *inverse* to x if

$$x \star y = e, \text{ and } y \star x = e.$$

Theorem 11.15. If $x \in X$ is an element in a monoid which has an inverse, then that inverse is unique.

Proof. Suppose both y and z were inverses to x. Then we can write

$$y = y \star e = y \star (x \star z) = (y \star x) \star z = e \star z = z.$$

So $y = z$. □

Definition 11.16. We say a monoid $(X, \star, e)$ is a *group* if each element admits an inverse.

Example 11.17. $(\mathbb{Z}, +)$ is a group with identity 0, and the inverse to x given by $-x$.

Example 11.18. $(\mathbb{N}, +)$ is not a group, since elements don't have inverses under addition.

Example 11.19. The nonzero rational numbers $(\mathbb{Q} \setminus \{0\}, \cdot)$ form a group under multiplication.

Example 11.20. The set Σ_n of permutations of the set $\{1, 2, \ldots, n\}$ form a group. This is a really important example – we call it the *symmetric group*.

Definition 11.21. If X is a set with n elements, then a *Latin square* for X is an $n \times n$ grid where each element $x \in X$ appears exactly once on each row and column.

Example 11.22. For $X = \{a, b\}$, there are two Latin squares for X, which are

a	b
b	a

b	a
a	b

Given a group, we can write out its multiplication table, sometimes called a *Cayley table*. These differ from monoids in the following super key way!

Proposition 11.23. If $(X, \star)$ is a group, then its Cayley table is a Latin grid.

Proof. We prove it for cols and see that . is similar. Recall each entry is row$\star$column.

In the column for y, we claim that x appears once. This is because we can look at the row corresponding to $z = x \star y^{-1}$. Then

$$z \star y = (x \star y^{-1}) \star y = x \star (y^{-1} \star y) = x \star e = x.$$

We claim x doesn't appear two (or more) times in the column. Indeed suppose x appeared in the z_1 and z_2 rows in the column corresponding to y. Then we have that

$$x = z_1 y = z_2 y.$$

Multiplying on the right by y^{-1}, we get

$$z_1 = z_1 y y^{-1} = z_2 y y^{-1} = z_2.$$

So $z_1 = z_2$, and x is only in that row! □

Suppose we want to classify group structures on a set with n elements, this lets us dramatically reduce from the number of binary operations n^{n^2} down to the number of Latin grids!

n	1	2	3	4	$\cdots$
number of binary operations, i.e. n^{n^2}	1	16	19,683	over 4 billion	$\cdots$
number of $n \times n$ Latin squares	1	2	12	576	$\cdots$

There is no known formula for the number of Latin squares in terms of n. We know it is at least $\geq \frac{(n!)^{2n}}{n^{n^2}}$.

Exercise 11.24. Let $X = \{a, b\}$ be a set with two elements. This set has sixteen possible binary operations[21] (here the entry is "row $\star$ column"):

	a	b
a	a	a
b	a	a

	a	b
a	a	a
b	a	b

	a	b
a	a	a
b	b	a

	a	b
a	a	a
b	b	b

	a	b
a	a	b
b	a	a

	a	b
a	a	b
b	a	b

	a	b
a	a	b
b	b	a

	a	b
a	a	b
b	b	b

	a	b
a	b	a
b	a	a

	a	b
a	b	a
b	a	b

	a	b
a	b	a
b	b	a

	a	b
a	b	a
b	b	b

	a	b
a	b	b
b	a	a

	a	b
a	b	b
b	a	b

	a	b
a	b	b
b	b	a

	a	b
a	b	b
b	b	b

How many of these are monoids? Are they commutative? Can you interpret any of them in terms of Boolean (true-false valued) logic?

Exercise 11.25. Define a binary operation on $\mathbb{Z}$ by $a \star b = a^b$. Is this a monoid? Why or why not?

Exercise 11.26. Let $B = \{\text{True}, \text{False}\}$ be the set of two truth values.

(1) Show that the operation AND (denoted $\wedge$) turns B into a commutative monoid. What is the unit?

[21] These are called the sixteen binary boolean operators.

(2) Show that the operation OR (denoted $\vee$) turns B into a commutative monoid. What is the unit?
(3) Are these the only monoid operations or are there more?
(4) Can you relate this to Problem 1?

Sometimes two Latin squares can be obtained from one another by *relabeling* the elements in the set. For instance, the Latin squares

a	b	c
b	c	a
c	a	b

and

b	c	a
c	a	b
a	b	c

are structurally the same – they differ by a relabeling of the elements.

Exercise 11.27. Write out a few (maybe not all 576) of the 4×4 Latin squares on the set $X = \{a, b, c, d\}$. Can you obtain all of them by *relabeling*?

12. Homomorphisms and isomorphisms

$\circ$	e	a	b	c	d
e	e	a	b	c	d
a	a	e	d	b	c
b	b	c	e	d	a
c	c	d	a	e	b
d	d	b	c	a	e

then $(ab)c = dc = a$ and $a(bc) = ad = c$. not a group!
As we have seen last time, the *Cayley table* of any group is a Latin square.
Let's write down some 4×4 Latin squares:

a	b	c	d
b	c	d	a
c	d	a	b
d	a	b	c

b	c	d	a
c	d	a	b
d	a	b	c
a	b	c	d

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

how are these different? How are they similar?

Definition 12.1. Let G and H be groups. A function

$$f \colon G \to H$$

is called a *homomorphism* if

$$f(g_1 \cdot g_2) = f(g_1) \cdot f(g_2)$$

for every $g_1, g_2 \in G$.

Theorem 12.2. Let

$$f \colon G \to H$$

be a group homomorphism. Then

(1) $f(1_G) = 1_H$, in other words f preserves the unit
(2) $f(g^{-1}) = f(g)^{-1}$ for all $g \in G$
(3) $f(g^k) = f(g)^k$ for any $k \geq 1$.

Proof. For the first one

$$f(1_G) \cdot f(1_G) = f(1_G \cdot 1_G) = f(1_G) = f(1_G) \cdot 1_H$$

Cancelling in H, we get $f(1_G) = 1_H$.

For the second, we see that

$$f(g^{-1})f(g) = f(g^{-1}g) = f(1_G) = 1_H.$$

Therefore $f(g^{-1})$ is the inverse to $f(g)$ in H, so $f(g^{-1}) = f(g)^{-1}$, since we proved last week that inverses are unique.

For the last point, we induct! Clearly the $k = 1$ case is true. Suppose we have proven it for k and want to see it for $k + 1$, then

$$f(g^{k+1}) = f(g \cdot g^k) = f(g) \cdot f(g^k) = f(g) \cdot f(g)^k = f(g)^{k+1}.$$

So we're done. □

Definition 12.3. A homomorphism which is bijective is called an *isomorphism*.

Let $G = \{1, g, h\}$ be a set with three elements, and consider the binary operation given by the Cayley table

	1	g	h
1	1	g	h
g	g	h	1
h	h	1	g

We claim this is a group. Clearly 1 is the identity, g and h are inverses, and we can verify that multiplication is associative.

We can also define a group

$$H = \{a, b, c\}$$

with the Cayley table

	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

Is this a group? Why or why not?

Proposition 12.4. The bijection

$$\begin{aligned} f \colon G &\to H \\ 1 &\mapsto a \\ g &\mapsto b \\ h &\mapsto c \end{aligned}$$

is an *isomorphism*.

Proof. todo □

Given two groups G and H like those above, we write $G \cong H$ to mean they are *isomorphic*.

Studying groups *up to isomorphism* means we only consider two groups to be different if they are *not isomorphic*.

Theorem 12.5. Up to isomorphism, there is only one group structure on a set with two elements.

Proof. We define $G = \{e, g\}$ to be a set with two elements, where e is the identity, and $g^2 = e$. It is clear this is a group.

Let $X = \{x_1, x_2\}$ be a set with two elements, and suppose it has a group structure. Since its Cayley table must be a Latin grid, there are only two possibilities for its Cayley table, namely:

	x_1	x_2
x_1	x_1	x_2
x_2	x_2	x_1

or

	x_1	x_2
x_1	x_2	x_1
x_2	x_1	x_2

In the first case, we observe that x_1 is the identity, so we claim that

$$\begin{aligned} f\colon X &\to G \\ x_1 &\mapsto e \\ x_2 &\mapsto g \end{aligned}$$

is a bijection. Indeed we check

$$\begin{aligned} f(x_1 \cdot x_1) &= f(x_1) = e = e \cdot e = f(x_1) \cdot f(x_1) \\ f(x_1 \cdot x_2) &= .... \end{aligned}$$

□

The existence of group isomorphisms means *it doesn't matter what we call the elements of our set*! The only thing that matters is how they multiply, i.e. how their Cayley tables look.

Example 12.6. Recall there are 16 logic gates, corresponding to binary operations on the set $B = \{T, F\}$. Since there are only two 2×2 Latin squares, and both give group structures, we have that exactly two of them give group structures. These correspond to the following Cayley tables, which we will also write as truth tables:

	T	F
T	F	T
F	T	F

which is

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

and

	T	F
T	T	F
F	F	T

which is

P	Q	$P \odot Q$
T	T	T
T	F	F
F	T	F
F	F	T

These notations $\oplus$ and $\odot$ are also called XOR and XNOR. In terms of basic operations, we see that $P \oplus Q$ means "P or Q but not both." In symbolic logic, this is:

$$P \oplus Q = (P \vee Q) \wedge \neg(P \wedge Q).$$

The exclusive nor operation XNOR is the negation of XOR. We can also think of it as "P and Q or not P and not Q." That is,

$$P \odot Q = (P \wedge Q) \vee (\neg P \wedge \neg Q).$$

These two groups are isomorphic, and since the unit must be sent to the unit under a group isomorphism, the function exhibiting the isomorphism is negation $\neg\colon B \to B$. The fact that negation is a group isomorphism is precisely a case of deMorgan's laws for exclusive or and exclusive nor:

$$\begin{aligned} \neg(P \oplus Q) &= \neg P \odot \neg Q \\ \neg(P \odot Q) &= \neg P \oplus \neg Q. \end{aligned}$$

Exercise 12.7. Fill out the Cayley tables:

	1	a	b	c	d
1	1	a	b	c	d
a	a		1	b	
b	b				
c	c				
d	d				

	1	a	b	c	d
1	1	a	b	c	d
a	a				
b	b		c	d	
c	c				
d	d				

It turns out that these give associative multiplications on the set $X = \{1, a, b, c, d\}$, hence they form groups (you can either trust me on this or show it yourself!). Are these two group structures on X isomorphic? If so, find an explicit isomorphism between them. If not, argue why not.

Exercise 12.8. Prove that up to isomorphism there is *only one group* with three elements.

Exercise 12.9. Up to isomorphism, how many groups with four elements are there?

Exercise 12.10.

(1) Define a group G, which has an identity element ϵ, and elements σ and τ which satisfy the relations

$$\sigma^3 = \epsilon$$
$$\tau^2 = \epsilon$$
$$\sigma\tau = \tau\sigma^2.$$

Argue (you don't have to prove this rigorously, but think about how you would prove it) that G has *six* elements, that is, show that the set

$$\left\{\epsilon, \sigma, \tau, \sigma^2, \tau, \tau\sigma, \tau\sigma^2\right\}$$

is closed under multiplication.

(2) Fill out the Cayley table for G:

	ϵ	σ	σ^2	τ	$\tau\sigma$	$\tau\sigma^2$
ϵ						
σ						
σ^2						
τ						
$\tau\sigma$						
$\tau\sigma^2$						

(3) Prove that G is isomorphic to the symmetric group Σ_3, whose Cayley table we wrote out on worksheet 9.

Exercise 12.11. Let $a, b \in G$ be elements in a group, and let $1 \in G$ be the identity element of the group

(1) If $a^4 = 1$ and $ab = ba^2$, show that $a = 1$
(2) If $a^6 = 1$ and $ab = ba^3$, show that $a^2 = 1$ and $ab = ba$
(3) If $a^6 = 1$ and $ab = ba^2$, show that $a^3 = 1$ and $aba = b$.
(4) If $(ab)^n = 1$ for some $n \geq 0$, show that $(ba)^n = 1$.

Exercise 12.12. If $fgh = 1$ in a group G, argue that $ghf = 1$. Is it true that $gfh = 1$?

Exercise 12.13. If $g \in G$ is an element in a group, show that $g^2 = 1$ if and only if $g^{-1} = g$.

13. SUBGROUPS

We saw on the homeworks some examples of groups living "inside" of other groups. Let's make this precise.

Definition 13.1. Let G be a group. A subset $H \subseteq G$ is called a *subgroup* if H is a group itself, with group operation defined by the group structure on G.

Example 13.2. If G is a group, then $G \subseteq G$ is a trivial subgroup of itself. Also the subset containing only the identity is a subgroup $\{1\} \subseteq G$.

Example 13.3. We have that $(\mathbb{Z}, +) \subseteq (\mathbb{Q}, +)$ is a subgroup.

Example 13.4. The even numbers form a subgroup of $(\mathbb{Z}, +)$. The odd numbers don't (because they're not closed under addition, *and* they don't have zero).

Example 13.5. We have that $(\{+1, -1\}, \cdot) \subseteq (\mathbb{Q} \smallsetminus \{0\}, \cdot)$ is a subgroup.

Example 13.6 (from HW5)**.** We have that

$$C_4 = \{\epsilon,\ (1\ 2\ 3\ 4),\ (1\ 3)(2\ 4),\ (1\ 4\ 3\ 2)\},\ \text{and}$$
$$K_4 = \{\epsilon,\ (1\ 2),\ (3\ 4),\ (1\ 2)(3\ 4)\}$$

are subgroups of Σ_4.

Theorem 13.7 (Subgroup test)**.** If G is a group, then a nonempty subset $H \subseteq G$ is a subgroup if and only if three things hold:

(1) H contains the unit, meaning $1_G \in H$
(2) H is closed under multiplication, meaning if $h_1, h_2 \in H \subseteq G$, then $h_1 \cdot h_2 \in H$.
(3) H contains inverses: if $h \in H$ then $h^{-1} \in H$, where h^{-1} means the inverse of h in G.

Proof. If H satisfies all the conditions, then it is closed under multiplication by (2), it has a unit by (1), has inverses by (3) and the group operation is associative since the group structure in G was associative.

Conversely, suppose H is a subgroup, we claim it satisfies all the conditions. For (1), we have that

$$1_H^2 = 1_H = 1_H \cdot 1_G,$$

so $1_H = 1_G$. Property (2) is part of the data of being a subgroup. For property (3) if h' is the inverse of h in H and h^{-1} is its inverse in G, then

$$h'h = 1_H = 1_G = h^{-1}h,$$

so by cancellation in G, we have that $h' = h^{-1}$. □

When H is finite, we have a much simpler test to be a subgroup!

Theorem 13.8 (Finite subgroup test)**.** Let G be a group, and $H \subseteq G$ a nonempty finite subset. Then H is a subgroup if and only if H is closed under the group operation.

Proof. The forwards direction is immediate, so we want to show that if H is closed under operation, then it is a subgroup. We will do this via the subgroup test — we will show that H contains the unit and contains inverses.

Let $h \in H$ be any element, and start taking powers of it:

$$\{h, h^2, h^3, h^4, \ldots\} \subseteq H.$$

Since H is finite, the set with the powers of H must be finite as well. This implies that $h^n = h^{n+m}$ for some $m, n \geq 1$. By cancellation in G, this means that $1 = h^m$. Since $h^m \in H$, this implies $1 \in H$.

Since $1 = h^m$, we have that $h(h^{m-1}) = 1$. In other words, $h^{m-1} = h^{-1}$ in G. Thus since $h^{m-1} \in H$, we have that $h^{-1} \in H$, so H is closed under taking inverses. □

Exercise 13.9. Determine all the subgroups of

$$(\mathbb{Z}/6, +) = \{0, 1, 2, 3, 4, 5\}.$$

We ask which subsets are closed under addition mod 6. There are two *trivial* cases, namely $\{0\}$ and the whole set. Are there any others?

Some other $H \subseteq \mathbb{Z}/6$ has to contain 0, and we notice that if it contains 1 it is closed under addition, so it contains $1 + 1 = 2$, and 3, and so on and we get the whole group. So let's assume it doesn't contain 1.

It could contain 2, and then it must also contain $2 + 2 = 4$. We claim that

$$\{0, 2, 4\}$$

is a subgroup!

Similarly, $\{0, 3\}$ is a subgroup. These are all the subgroups.

14. Groups via generators and relations

One elegant way to present groups is to say they are *generated* by elements and relations.

Definition 14.1. A group G is *generated* by some elements $a_1, a_2, \ldots, a_n \in G$ if every element in G can be expressed as a product of the a_i's and a_i^{-1}'s.

That is, we write

$$G = \langle \text{generators} | \text{relations} \rangle.$$

If you go to the group theory wiki to learn anything, this is how you'll see it presented!

Example 14.2. We can let

$$G = \left\langle a, b \colon a^3 = b^2 = 1,\ ab = ba^2 \right\rangle.$$

This means take the group given by all the powers of a, all the powers of b, all their inverses, and all products of stuff in them. The relations let us simplify long expressions in a's and b's. For instance

$$a^5 b^{-7} a^3 b a b^{-1}$$

can be rewritten using the relations as

$$\begin{aligned} a^3 a^2 b^{-6} b^{-1} a^3 b a b^{-1} &= 1 a^2 (b^2)^{-3} b^{-1} 1 b a b^{-1} \\ &= a^2 b^{-1} b a b^{-1} \\ &= a^2 a b^{-1} \\ &= a^3 b^{-1} \\ &= b^{-1} \\ &= b. \end{aligned}$$

We can prove that G is, as a set, equal to

$$\{1, a, b, a^2, ba, ba^2\}.$$

Definition 14.3. If a group has a single generator, it is called a *cyclic group*.

Example 14.4. We can consider the group

$$C_6 = \left\langle g \colon g^6 = 1 \right\rangle.$$

This has as elements $\left\{1, g, g^2, g^3, g^4, g^5\right\}$.

We can check that

$$\begin{aligned} \mathbb{Z}/6 &\to G \\ n &\mapsto g^n \end{aligned}$$

is a group isomorphism!

Notation 14.5. We will write C_n for the group

$$C_n = \langle g \colon g^n = 1 \rangle .$$

This is called a *cyclic group of order n*.

Exercise 14.6. Consider the group

$$G = \left\langle a, b \colon a^2 = b^2 = 1, ab = ba \right\rangle .$$

Argue that $G \cong K_4$.

Exercise 14.7. Let D_4 be the group

$$D_4 = \left\langle x, a \colon a^4 = x^2 = e,\ xax^{-1} = a^{-1} \right\rangle .$$

Can you list all its subgroups? What are they isomorphic to?

Exercise 14.8. What are all the subgroups of C_{12}?

Exercise 14.9. Show the finite subgroup test fails if the subset is infinite – that is, find a group G and an infinite subset $H \subseteq G$ which is closed under the group operation but isn't a group.

Exercise 14.10. Let H, K be subgroups of a group G. Show that $H \cap K$ is also a subgroup of G.

Exercise 14.11. Let $H \subseteq G$ be a subgroup, and let $g \in H$. Define

$$gHg^{-1} := \left\{ ghg^{-1} \colon h \in H \right\} .$$

Show that gHg^{-1} is a subgroup of G.

15. Orders of elements and groups

Definition 15.1. If G is a finite group, we say its *order* is its size. If G is an infinite group, we say it has *infinite order*.

Unfortunately the word "order" is overloaded – we use order to talk both about groups as well as their elements.

Definition 15.2. If $g \in G$, its *order*, denoted $o(g)$, is the smallest integer $k \geq 1$ so that $g^k = 1$, unless no such integer exists, in which case we say $o(g) = \infty$.

Example 15.3. The order of the unit element is always 1.

Example 15.4. If $o(g) = 2$ then $g^2 = 1$, meaning that $g = g^{-1}$. An element has order two if and only if it is its own inverse.

Example 15.5. Consider the group

$$\{1, -1, i, -1\},$$

under multiplication of complex numbers. Then $o(1) = 1$, and $o(-1) = 2$, and $o(i) = o(-i) = 4$.

Example 15.6. In $(\mathbb{Z}, +)$, the order of 1 is infinite, since $1 + \ldots + 1$ will never equal zero.

Example 15.7. If $\sigma \in \Sigma_n$ is a cycle of the form

$$\sigma = (a_1\ a_2\ \cdots\ a_r),$$

then $o(\sigma) = r$.

Theorem 15.8. If $g \in G$ has order n, then

(1) $g^k = 1$ if and only if $n \mid k$
(2) $g^a = g^b$ if and only if $a \equiv b \pmod{n}$
(3) All the elements $\{1, g, g^2, \ldots, g^{n-1}\}$ are distinct, and they form a cyclic subgroup of G.

Proof. For the first point, if $n \mid k$ then $k = mn$ for some m, so $g^k = g^{mn} = (g^n)^m = 1^m = 1$. Conversely if $g^k = 1$, we use the division algorithm to write $k = qn + r$ for some $0 \leq r < n$. Then

$$1 = g^k = g^{qn+r} = (g^n)^q g^r.$$

So $g^r = 1$. If $r > 0$ then this contradicts n being the order of g. So we must have $r = 0$ and therefore $k \mid n$.

For the second part, if $g^a = g^b$, then $g^{a-b} = 1$, hence $n \mid (a - b)$ meaning that $a \equiv b \pmod{n}$, and vice versa.

Finally the division algorithm lets us show the final part. We can write any g^k as $g^{qn+r} = g^r$ where $0 \leq r \leq n - 1$. □

Corollary 15.9. Let $g \in G$, and let $\langle g \rangle \subseteq G$ be the cyclic subgroup generated by g. Then $|\langle g \rangle| = o(g)$. That is, this group is cyclic of order equal to the order of g.

This corollary is (probably) the reason for the conflation between the order of a group and the order of an element.

Question 15.10. If $\sigma \in \Sigma_n$ is a permutation, what is its order?

Example 15.11. If σ is an n-cycle, it has order n.

Example 15.12. What is the order of $(1\ 2\ 3)(4\ 5) \in \Sigma_5$?

Question 15.13. If $g, h \in G$ have orders $o(g) = a$ and $o(h) = b$, what is the order of gh?

In general this is quite hard to answer, and there's no general formula, it really depends on the context!

Definition 15.14. We say g and h *commute* if $gh = hg$.

Example 15.15. Suppose g and h commute, and $o(g) = 2$ and $o(h) = 3$. Then

$$\begin{aligned}(gh)^2 &= g^2h^2 = h^2\\ (gh)^3 &= g^3h^3 = g\\ (gh)^4 &= g^4h^4 = h\\ (gh)^5 &= g^5h^5 = gh^2\\ (gh)^6 &= g^6h^6 = 1.\end{aligned}$$

So $o(gh) = 6$.

Example 15.16. In Σ_3, we have that $o((1\ 2)) = 2$ and $o((1\ 2\ 3)) = 3$. However

$$(1\ 2)(1\ 2\ 3) = (2\ 3),$$

which has order two. This is because these two elements *don't commute*:

$$\begin{aligned}(1\ 2)(1\ 2\ 3) &= (2\ 3)\\ (1\ 2\ 3)(1\ 2) &= (1\ 3).\end{aligned}$$

If cycles are *disjoint*, we have a nicer answer.

Definition 15.17 (lcm)**.**

(1) Given two positive natural numbers a and b, their *least common multiple*, denoted $\text{lcm}(a, b)$, is the smallest positive integer k which is a multiple of a and b, meaning the smallest k for which $a \mid k$ and $b \mid k$.
(2) Given n positive natural numbers $d_1, \ldots, d_n$, their least common multiple $\text{lcm}(d_1, \ldots, d_n)$ is the smallest positive integer k for which $d_i \mid k$ for each $1 \leq i \leq n$.

Proposition 15.18. If $d_1, \ldots, d_n > 0$, and $k \in \mathbb{N}$ satisfies $d_i \mid k$ for each i, then $\text{lcm}(d_1, \ldots, d_n) \mid k$.

Proof. If $\text{lcm}(d_1, \ldots, d_n) = k$ then we're done. If not, then $\text{lcm}(d_1, \ldots, d_n) < k$, since lcm was the *least* common multiple. By the division algorithm we get

$$k = q \cdot \text{lcm}(d_1, \ldots, d_n) + r,$$

where $0 \leq r < \text{lcm}(d_1, \ldots, d_n)$. We can rewrite

$$r = k - q \cdot \text{lcm}(d_1, \ldots, d_n).$$

Since $d_i \mid k$ and $d_i \mid \text{lcm}(d_1, \ldots, d_n)$ for each i, this implies $d_i \mid r$. So r is a common multiple of the d_i's, and $r < \text{lcm}(d_1, \ldots, d_n)$. This implies that $r = 0$. □

Theorem 15.19. If $\sigma \in \Sigma_n$ has a disjoint cycle decomposition as

$$\sigma = \alpha_1 \alpha_2 \cdots \alpha_r,$$

with α_i a d_i-cycle, then $o(\sigma) = \text{lcm}(d_1, \ldots, d_r)$.

Proof. Write $\ell = \text{lcm}(d_1, \ldots, d_n)$ for simplicity. We claim that $o(\sigma) \mid \ell$ and $\ell \mid o(\sigma)$, and this will prove they are equal (since they are both positive integers).

To see that $o(\sigma) \mid \ell$, we will argue that $\sigma^\ell = 1$, after which we can use the previous theorem. Since each $d_i \mid \ell$, we can write $\ell = q_i d_i$ for some $q_i \in \mathbb{Z}$. Then

$$\sigma^\ell = (\alpha_1 \cdots \alpha_r)^\ell.$$

Since the α_i's commute, we can shuffle everything around and write the above as

$$\begin{aligned}
&= \alpha_1^\ell \cdots \alpha_r^\ell \\
&= \alpha_1^{q_1 d_1} \alpha_2^{q_2 d_2} \cdots \alpha_r^{q_r d_r} \\
&= (\alpha_1^{d_1})^{q_1} (\alpha_2^{d_2})^{q_2} \cdots (\alpha_r^{d_r})^{q_r} \\
&= 1^{q_1} \cdots 1^{q_r} \\
&= 1.
\end{aligned}$$

This proves $o(\sigma) \mid \ell$.

To show that $\ell \mid o(\sigma)$, it suffices to see that $o(\sigma)$ is a multiple of each d_i, since the lcm of the d_i's will divide any multiple of all the d_i's by that previous proposition. Again by the theorem on orders, we want to show that $\alpha_i^{o(\sigma)} = 1$ for any i and we will be done. We will argue for $i = 1$ and see it is true for the remaining ones by a similar argument.

If k is fixed by α_1, then it is also fixed by $\alpha_1^{o(\sigma)}$. If k is not fixed by α_1, then it *is* fixed by all the other α_i's. So we see that

$$\begin{aligned}
k = \sigma^{o(\sigma)}(k) &= \alpha_1^{o(\sigma)} \cdots \alpha_r^{o(\sigma)}(k) \\
&= \alpha_1^{o(\sigma)}(k).
\end{aligned}$$

Hence $\alpha_1^{o(\sigma)}$ fixes k no matter what, and therefore $\alpha_1^{o(\sigma)}$ is the identity. □

Remark 15.20. We really needed that we were talking about *permutations* here. This type of argument doesn't hold in general for groups (i.e. if $g_1, \dots, g_n$ pairwise commute in some group G, it is *not* true that the order of their product is the lcm of the orders of the g_i's. This is kind of a subtle question in group theory and requires a bit more machinery than we currently have to solve it even in the context of abelian groups).

Exercise 15.21. For $g \in G$ a group element, show that $o(g) = o(g^{-1})$.

Exercise 15.22. Compute the order of the following permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 4 & 8 & 7 & 2 & 3 & 5 & 6 & 1 \end{pmatrix}.$$

Exercise 15.23. Let $g \in G$ be a group element. If $o(g) = n$, and $d \mid n$ for $d \geq 1$, then show that $o(g^d) = \frac{n}{d}$.

Exercise 15.24. Let $f \colon G \to H$ be a homomorphism of groups.

(1) If $g \in G$ has order n, what can you say about the order of $f(g)$?
(2) Suppose f is an isomorphism. What can we say about the order of $f(g)$ in terms of the order of g?

16. Dihedral groups

The origin of group theory is the study of *symmetry*. As an example, consider a square. We can ask about the *symmetries* of the square. Let's list all the symmetries:

- ▷ rotational
- ▷ reflective

How are these related? We can start to give them names, for instance rotating the square clockwise 90° we can call "r." Flipping the square over a vertical axis we could call s. We can start to see some relations:

- ▷ rotating counterclockwise undoes r, so we could call it r^{-1}.
- ▷ rotating 270° clockwise is the same as rotating 90° counterclockwise, which we could phrase as an equality $r^3 = r^{-1}$. This is equivalent to saying $r^4 = 1$
- ▷ flipping twice over the vertical axis doesn't do anything, so $s^2 = 1$.
- ▷ rotating and flipping is the same as flipping then rotating in the other direction, so $sr = r^{-1}s$.

Altogether, we can package this information as a *group* with generators and relations:

$$D_4 = \left\langle r, s \colon r^4 = s^2 = 1,\ sr = r^{-1}s \right\rangle.$$

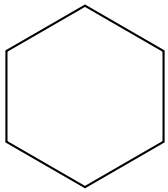
We call this the *dihedral group*.

[todo - add some more about dihedral groups]

What are some subgroups of this? Can we visualize them?

- ▷ $C_4 \cong \langle r \rangle$, just the rotational symmetries
- ▷ $C_2 \cong \langle s \rangle$, just the swapping symmetry
- ▷ others... (ten subgroups in total)

Exercise 16.1. Write out the group of symmetries of a regular hexagon:



Verify you get a group of order 12.

Exercise 16.2. Call the group from the previous exercise D_6. Can you write it as a subgroup of Σ_6? How?

Exercise 16.3. Give a presentation for the group of symmetries of a regular n-gon.

Exercise 16.4. Call D_n the group of symmetries of the regular n-gon. For which k and n is D_k a subgroup of D_n?

Definition 16.5. A *transposition* is a 2-cycle in a symmetric group. For instance (3 4) or (1 6) are transpositions in Σ_n.

Exercise 16.6.

(1) Write D_4 as a subgroup of Σ_4 (similarly to as in Exercise 16.2).
(2) For each of the eight elements in D_4, write it as a permutation, then write that permutation as a *product of transpositions*! For example,
$$(1\ 2\ 3\ 4) = (1\ 4)(1\ 3)(1\ 2).$$
(3) Half of the elements in D_4 flip the square over and half of them don't. Divide the eight elements in D_4 into these two categories.
(4) What is the connection between these two types of elements in D_4 and their factorizations into transpositions? Try factoring some elements for D_5 and see if your guess is right. Formulate this into a precise mathematical conjecture.

17. Cayley's theorem

The goal today is to prove the following:

Theorem 17.1 (Cayley's theorem)**.** Every group G of order n is isomorphic to a subgroup of Σ_n.

We'll do this over a few exercises.

Exercise 17.2. For any set X, denote by Σ_X the set of bijections from X to itself:
$$\Sigma_X := \{f\colon X \to X \mid f \text{ is a bijection}\}.$$
Show that Σ_X is a group under function composition.

Exercise 17.3. If $\sigma\colon X \to Y$ is a bijection between two sets, show that σ induces a group isomorphism
$$\begin{aligned}\Sigma_X &\to \Sigma_Y\\ f &\mapsto (\sigma \circ f \circ \sigma^{-1}).\end{aligned}$$

Exercise 17.4. If G is a group, and $g \in G$ is any fixed element, we can define
$$\begin{aligned}\mu_g\colon G &\to G\\ x &\mapsto gx.\end{aligned}$$
Show this is a bijection on G.

Exercise 17.5. Show that the assignment
$$\begin{aligned}G &\to \Sigma_G\\ g &\mapsto \mu_g\end{aligned}$$
is an injective group homomorphism.

Exercise 17.6 (Proving Cayley's theorem)**.** Let G be a finite group of order n. Argue that $\Sigma_G \cong \Sigma_n$, and therefore that G is isomorphic to a subgroup of Σ_n.

18. Orbit-stabilizer

Let X be a set and G be a group. We say that X is a *G-set* if X has symmetry under the group G. Let's make this explicit.

Definition 18.1. A *group action* of G on a set X is a group homomorphism $G \to S_X$.

Explicitly it is an assignment of $g \cdot x \in X$ for every $g \in G$, so that

(1) $1 \cdot x = x$ for all $x \in X$
(2) $g \cdot (h \cdot x) = (g \cdot h) \cdot x$

Definition 18.2. A *G-set* is a set equipped with a G-action.

Example 18.3. The four vertices $V = \{v_1, v_2, v_3, v_4\}$ of a square form a D_4-set. The group action $D_4 \to S_V$ is exactly the injection from $D_4 \to \Sigma_4$ that we found.

Definition 18.4. If X is a G-set, the *orbit* of a point $x \in X$ is

$$G \cdot x := \{g \cdot x \mid g \in G\} \subseteq X.$$

That is, these are all the points that can be reached from $x \in X$.

Given a G-set X and an element $x \in X$, we can look at the group elements that send x to itself.

Definition 18.5. The *stabilizer* of an element $x \in X$ is

$$\mathrm{Stab}_G(x) := \{g \in G \mid g \cdot x = x\} \subseteq G.$$

Proposition 18.6. If X is a G-set and $x \in X$ is a point, then

$$\mathrm{Stab}_G(x) \subseteq G$$

is a subgroup.

Proof. Clearly $1 \in \mathrm{Stab}_G(x)$ since $1 \cdot x = x$. To see that $\mathrm{Stab}_G(x)$ is closed under taking inverses, suppose$g \in G$. Then $g \cdot x = x$. Applying g^{-1} on either side, we get

$$g^{-1}g \cdot x = g^{-1}x.$$

Since $g^{-1}gx = 1x = x$, we have that $g^{-1}x = x$, hence $g^{-1} \in \mathrm{Stab}_G(x)$. Finally to see that $\mathrm{Stab}_G(x)$ is closed, suppose that $g_1, g_2 \in \mathrm{Stab}_G(x)$. Then

$$(g_1 \cdot g_2) \cdot x = g_1 \cdot (g_2 \cdot x) = g_1 \cdot x = x.$$

So $g_1 g_2 \in \mathrm{Stab}_G(x)$. □

We'll now state a huge result which will help us figure out the sizes of automorphism groups.

Theorem 18.7 (Orbit-stabilizer theorem)**.** Let G be a finite group and let X be a G-set, and pick any element $x \in X$. Then

$$\#G = \#\mathrm{Stab}_G(x) \cdot \#G \cdot x.$$

That is, the order of G is the size of the stabilizer times the size of the orbit.

Example 18.8 (dihedral)**.** Let G be the group of symmetries of the square. Suppose we didn't know anything about it. We can pick a vertex v for instance. There is one symmetry which preserves the vertex, which is the diagonal symmetry. So $\mathrm{Stab}_G(v) \cong C_2$. We can get from any vertex to any other vertex, so $\#G \cdot v = 4$. By the orbit stabilizer theorem, we have that

$$\#G = \#\mathrm{Stab}_G(v) \cdot \#Gv = 2 \cdot 4 = 8.$$

So we know that the group of symmetries has order 8, even before writing down any presentation of it.

Example 18.9. Note that Σ_n acts on the set $X = \{1, \dots, n\}$. The stabilizer $\mathrm{Stab}_{\Sigma_n}(n)$ is all the elements in Σ_n which leave n fixed. In particular we have that $\mathrm{Stab}_{\Sigma_n}(n) \cong \Sigma_{n-1}$. The orbit of n is the entire set X, since there is always a permutation taking n to any other number. Hence by the orbit stabilizer, we have

$$|\Sigma_n| = |\mathrm{Stab}_{\Sigma_n}(n)||X| = (n-1)!n.$$

Example 18.10. Let $V = \{v_1, v_2, v_3, v_4\}$ be the four vertices of the tetrahedron, and let G be the group of symmetries of the tetrahedron. We *don't know the size of* G, but we can try to figure out using the orbit-stabilizer theorem! Note that we can always rotate the tetrahedron around so any vertex goes to any other vertex. Therefore $Gv_1 = V$, which has four elements. What is the size of $\mathrm{Stab}_G(v_1)$? That is, what symmetries fix the vertex v_1? If we don't turn the tetrahedron inside out, we can only rotate, so we get $\mathrm{Stab}_G(v_1) \cong C_3$. Altogether we get that

$$|G| = |\mathrm{Stab}_G(v_1)| \cdot |Gv_1| = 4 \cdot 3 = 12.$$

This tells you the group of symmetries has order 12. This helps us narrow down what it is! We now want to write down some generators, record relations, and we know we will have finished when we have enough relations to describe a group of order 12.

Example 18.11. We could also have computed this by using edges (or faces) of the tetrahedron. There are six edges $E = \{e_1, \dots, e_6\}$. Again we claim that $Ge_1 = E$, so there is one orbit of size six. Given any one edge, there is only a single symmetry which sends it to itself, namely rotation by 180° about the center of the edge. Hence we claim that $\mathrm{Stab}_G(e_1) \cong C_2$. Altogether by orbit-stabilizer we get

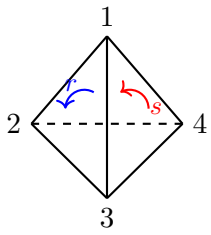
$$|G| = |\mathrm{Stab}_G(e_1)||E| = 2 \cdot 6 = 12.$$

This is the same answer, because it has to be, but it's still cool we could get it in different ways!

19. Symmetry groups of Platonic solids

For the tetrahedron, the orbit-stabilizer theorem gives us 12 symmetries. We can rotate around any vertex 120° clockwise or counter-clockwise, yielding 8 symmetries, and we can flip the tetrahedron across any edge (which also flips across the opposite edge), yielding 3 more symmetries. Together with the identity, we get all 12.

Clearly this group isn't cyclic, since none of these symmetries have order 12, so we need at least two of these symmetries to generate the group. Consider the two rotations r and s as pictured:



For the tetrahedron, we could pick the rotations r and s, from which we clearly have the relations $r^3 = s^3 = 1$, we also see that

$$s^2 r = (1\ 4\ 3)(1\ 2\ 3) = (1\ 2)(3\ 4),$$

which is one of these order two symmetries. By rotating, we get that all pairs of edge flips are in the subgroup generated by r and s.

Can we get the other rotations from r and s? The answer is yes – to rotate along the 234 plane we apply

$$rs = (1\ 2\ 3)(1\ 3\ 4) = (2\ 3\ 4),$$

and to rotate along the 124 plane we do

$$sr = (1\ 3\ 4)(1\ 2\ 3) = (1\ 2\ 4).$$

Hence G is generated by r and s. We see that $(rs)^3 = 1$ and that s^2r has order two, hence we claim we get a presentation:

$$G \cong \left\langle r, s \mid r^3 = s^3 = (rs)^3 = (s^2r)^2 = 1 \right\rangle.$$

One way to do this is using a computer, e.g. using GAP

```
gap> F2:=FreeGroup("a","b");
<free group on the generators [ a, b ]>
gap> T:= F2/[F2.1^3, F2.2^3, (F2.1*F2.2)^3, (F2.2*F2.2*F2.1)^2];
<fp group on the generators [ a, b ]>
gap> Order(T);
12
```

This is a way to check we truly have enough relations. How does this work though – i.e. how is GAP verifying the order of a given group? Different methods work for different groups, but at least for finitely presented groups, the method carried out is as follows:

(1) pick an element, take powers of it until we get the identity, so we have determined its order. It gives us some cyclic subgroup of G, which we may as well call C_n.
(2) determine the size of G by finding the index of C_n in G. This is done via the *Todd-Coxeter coset enumeration algorithm*, first outlined in the 1930's.

20. Platonic solids

[see notes on Canvas]

21. Beyond Platonic solids

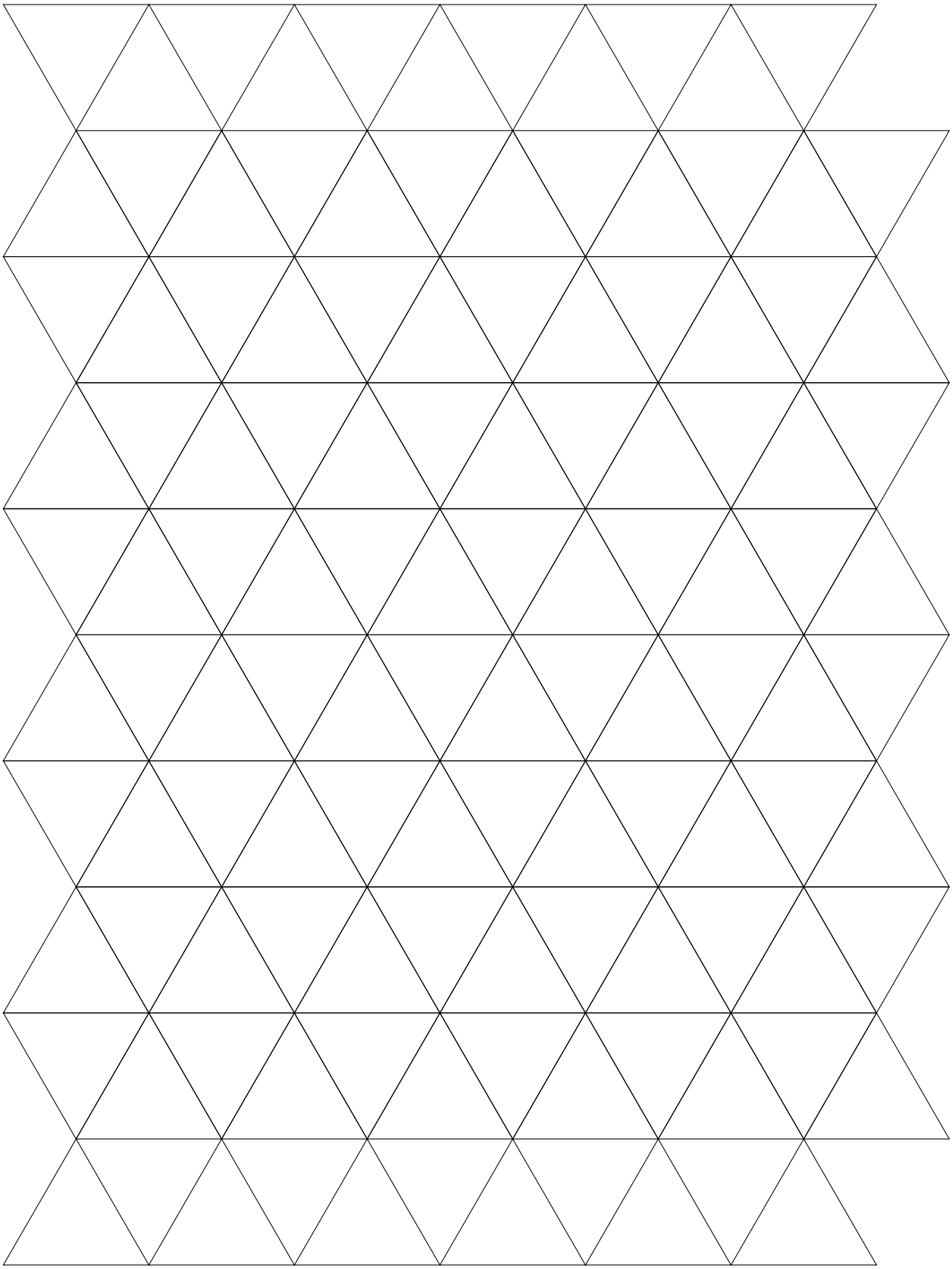
For building Platonic solids, we had a few rules:

(1) we are only allowed to build *convex* shapes
(2) we are only allowed to use *regular* polygons
(3) every vertex must look the same
(4) we can only use the same polygon (e.g. only triangles, or only squares..)

When we did this, we found five possibilities, which are the Platonic solids! We are now curious about weakening our rules a little bit and seeing what else is out there. In particular, what happens if we drop that last rule and we are now allowed to use multiple polygons? Again we will only build convex shapes, using regular polygons, and all of whose vertices look the same (we call this condition *vertex-regular*). So a precise question is:

Question 21.1. What are all the convex, vertex-regular polyhedra, with regular polygons as faces?

On the following pages, we have some regular polyhedra we can cut out and build shapes with. All the side lengths are 1 inch so that we can glue them.



In trying to build these, we can just start sticking shapes together and gluing, and we bump into that idea of *angle defect* that we discussed when building Platonic solids. We notice that:

- ▷ we need at least three polygons meeting at any vertex,
- ▷ the sum of their interior angles has to be strictly less than 360°.

Just with these rules, there are a lot of possibilities. In particular, once you determine how each vertex should look, how do you know how many vertices the final shape will have? The following theorem gives us an answer.

Theorem 21.2 (Descartes' Theorem)**.** Consider a convex, vertex regular[22] polyhedron with v vertices and an angle defect of θ degrees. Then we have that

$$v \cdot \theta = 720^\circ.$$

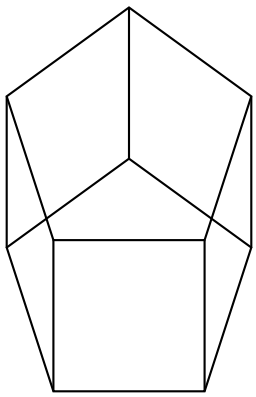
We'll prove this theorem later, since we can provide a really elegant intuitive proof at the cost of assuming another theorem we'll state soon. First though, let's reality check this against what we know about Platonic solids!

solid	**number of vertices v**	**angle defect θ**
tetrahedron	4	180°
octahedron	6	120°
cube	8	90°
icosahedron	12	60°
dodecahedron	20	36°

Indeed Descarte's Theorem gels with what we know – the product of columns two and three above is always equal to 720°.

21.1. **Prisms and antiprisms.** In trying to build these shapes, you may have stumbled across a partial answer to Question 21.1 – there are *infinitely many*!

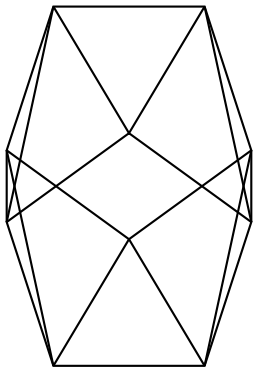
Here is a recipe to build a polyhedron using two n-gons for any $n \geq 3$ – you can let each vertex touch one n-gon and two squares. When we do this (for example if $n = 5$, we get the following shape):



This is called a *prism*. We can build a n-gonal prism by taking two n-gons and connecting their edges by squares.

Another thing we could do with any n-gon is let it touch three triangles at any vertex. Carrying out this construction with $n = 5$ we get the following shape:

[22]There is a more general version of this theorem that doesn't require convexity or vertex-regularity, but we won't touch on it in this class.



This is called a *pentagonal antiprism*. We can form an n-gonal antiprism for any $n \geq 3$.

Example 21.3. A square prism is a cube.

Example 21.4. A triangular antiprism is an octahedron.

For posterity, let's record the vertices and angle defects for prisms and antiprisms.

solid	**number of vertices** v	**angle defect** θ
n-gonal prism	$2n$	$360/n^\circ$
n-gonal antiprism	$2n$	$360/n^\circ$

We still haven't answered Question 21.1, but we now know enough to ask a modified question:

Question 21.5. Not counting prisms and antiprisms, are there infinitely many vertex-regular convex polyhedra with regular polygonal faces?

We notice (and could prove) that if there were infinitely many, we would have to be able to use n-gons for arbitrarily large n. We will prove that this can't happen, and therefore there are only *finitely many* of these polyhedra we are hunting for which aren't prisms or antiprisms.

22. Archimedean solids

Definition 22.1. We say a polyhedron with symmetry group G is *vertex-transitive* if for any two vertices v_1 and v_2, there is a symmetry $g \in G$ so that $gv_1 = v_2$.

Note that vertex-transitivity implies vertex-regularity, but not vice versa (we'll see an example in a bit).

Definition 22.2. A polyhedron is called an *Archimedan solid* if it is convex, vertex-transitive, and all its faces are regular polygons.

These are listed as follows:

name	vertices	edges	faces	angle defect
Truncated Tetrahedron	12	18	8	60°
Cuboctahedron	12	24	14	60°
Truncated Cube	24	36	14	30°
Truncated Octahedron	24	36	14	30°
Rhombicuboctahedron	24	48	26	30°
Truncated Cuboctahedron	48	72	26	15°
Snub Cube	24	60	38	30°
Icosidodecahedron	30	60	32	24°
Truncated Dodecahedron	60	90	32	12°
Truncated Icosahedron	60	90	32	12°
Rhombicosidodecahedron	60	120	62	12°
Truncated Icosidodecahedron	120	180	62	6°
Snub Dodecahedron	60	150	92	12°

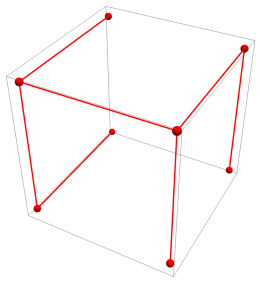
23. EULER'S FORMULA

Theorem 23.1 (Euler's formula)**.** Let P be any convex polyhedron[23] with V vertices, E edges and F faces. Then

$$V - E + F = 2.$$

This can be proven in a variety of ways. Here is a proof due to von Staudt in 1847, see [Cox48, 1.6].

Proof sketch. Imagine one vertex of the polygon is the heart, and it wants to pump blood to every other vertex. Label some edges in red so that every vertex receives exactly one path from the "heart vertex," and do it in such a way that no cycles are created.[24] The number of red edges is equal to $V - 1$:



Now put a blue dot on the center of every face, and imagine blood needs to be pumped between these dots, but the blue paths *can't cross the red edges*:

It turns out to be possible to construct a similar network of paths which is connected and has no cycles. Note that each path crosses a *unique edge* on P which hasn't been colored red, and moreover every edge in P must be either colored red or crossed by a blue edge. The number of blue edges we drew is

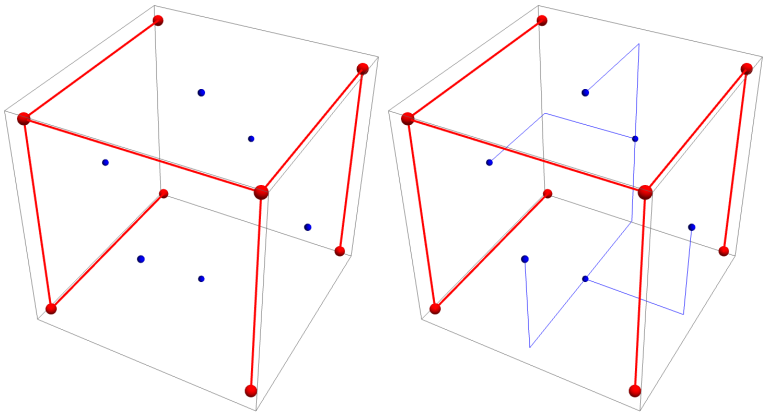
$$F - 1.$$

Since the blue and red edges together are in bijection with the edges of P, we get that

$$E = (V - 1) + (F - 1),$$

[23] There is a more general version where we need not require the polyhedra to be convex, they just have to be "like a sphere" in a sense that can be made precise using topology.

[24] The graph theory lingo is that we have made a *tree* among the vertices and edges.



and rearranging we get Euler's formula. □

We can now prove Descarte's theorem using Euler's formula.

Theorem 23.2 (Descarte's theorem)**.** Let P be any convex polyhedron.[25] Then the sum of the angle defects over the vertices of P is $720°$.

Proof. Pick any face on P. It is some n-gon, so we get that

$$\sum_v \text{interior angle at } v = (n-2)180,$$

since the sum of interior angles on an n-gon is always $(n-2)180$. We are going to rearrange the formula above to get:

$$0 = 360 - n180 + \sum_v \text{interior angle at } v.$$

We now interpret this in an odd way – we think about the face itself as contributing 360, each of the n edges as contributing 180, and each vertex as contributing its interior angle at the face under consideration. Let's now take this quantity, and add each of these up for each face. We get

$$0 = 360F - 360E + \sum_v (360 - \theta(v)),$$

where $\theta(v)$ is the angle defect at the vertex v. Rearranging, we get

$$360F - 360E + 360V = \sum_v \theta(v).$$

Applying Euler's formula on the left, we get that

$$720 = \sum_v \theta(v).$$

□

Aside from providing an elegant proof of Descartes' theorem, Euler's formula has some immediate applications. The following we know from

Proposition 23.3. Suppose a sphere is covered in triangles, so that r triangles meet at any one vertex. Then $r < 6$.

[25] Or more generally any connected polyhedron with $V - E + F = 2$.

Proof. We use some number F of triangles, the number of edges is $3F/2$ (immediately we need F to be even), and the number of vertices is $3F/r$ for whatever r is. Altogether, by Euler's formula we get

$$F - \frac{3F}{2} + \frac{3F}{r} = 2,$$

or in a slightly different form

$$\frac{6F}{r} = 4 + F.$$

What are possible values for r?

- ▷ If $r = 2$, this is technically possible, so long as we stretch our definition of what a *triangle* is.
- ▷ If $r = 3$, then $2F - 3F/2 = 2$, hence we get $F = 4$, and we have a tetrahedron.
- ▷ If $r = 4$, then we get $\frac{3}{2}F = 4 + F$, giving $F = 8$, and we get an octahedron.
- ▷ If $r = 5$ we get $\frac{6}{5}F = 4 + F$, so $F = 20$, and we get a dodecahedron.
- ▷ If $r = 6$, we get $F = 4 + F$, which is impossible. Indeed for $r \geq 6$, we have that $\frac{6F}{r} = 4 + F$ has no integral solutions in F.

□

Proposition 23.4. A sphere can't be tiled with hexagons, meeting r at a vertex.

Proof. Similar to the above, we would get

$$F - \frac{6F}{2} + \frac{6F}{r} = 2,$$

which after rearranging gives

$$\frac{3F}{r} = 1 + F$$

If $r \geq 3$ this has no solutions in F. If $r = 2$, then we get $\frac{3}{2}F = 1 + F$, which says $F = 2$. □

Exercise 23.5. Using Euler's formula,

(1) Prove that you can't tile a sphere using hexagons meeting three at every vertex.
(2) Suppose you want to tile a sphere using as many hexagons as you like and a few pentagons. What is the *minimum* number of pentagons you could use?

Exercise 23.6. Draw some polygons covering the surface of a donut.

(1) Is it still true that $V - E + F = 2$?
(2) Try drawing a different collection of polygons covering the surface of a donut. What is $V - E + F$?

Exercise 23.7 (Bonus)**.** We call the surface of a donut a *torus*.

(1) What is the correct analogue of Euler's formula for a torus?
(2) What is the correct analogue of Descarte's theorem on a torus?

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